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GUIDANCE SYSTEM ERROR STUDY PROGRAM
STATISTICAL MEASURES OF PERFORMANCE

Contract NAS8-11381

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FOREWORD

A principal objective of the Guidance System Error Study Program, Contract NAS8-11381, is to establish meaningful statistical measures of system performance. This report presents the results obtained in fulfillment of the above objective. The investigation was performed by LMSC/HREC for the Astrionics Laboratory of the George C. Marshall Space Flight Center.

SUMMARY

Basic concepts of probability theory are briefly reviewed and applied to the problem of determining meaningful measures of performance. Procedures for computing the equi-probability error volume, its orientation, and lengths of the principal semi-axes are presented. A closed form solution for the probability parameter associated with the equiprobability ellipsoid is derived.

Computer programs for determining the probability of hitting typical target windows are formulated. These target windows are three dimensional where the dimensions may apply to either the marginal distribution of position errors or the marginal distribution of velocity errors.

Finally, simplified statistical measures of performance are discussed. An equivalent spherical probable error is defined and a conservative approximation to the generalized variance is derived.

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1.0 INTRODUCTION

One objective in the formulation of the Guidance Error Study (Contract NAS8-11381) is to determine meaningful statistical measures of vehicle performance. Performance errors consist of position and velocity deviations from a nominal trajectory; these errors are due to hardware error sources in the vehicle's guidance system.

The central program of the guidance error computer program (developed under contract NAS8-11381) determines the covariances and mean values of the performance errors. Additional measures of vehicle performance may be determined on the basis of this data. These additional measures are more meaningful than the individual performance errors since they directly describe mission success criteria. A familiar example of a meaningful statistical measure that may be obtained from knowledge of the covariances of position errors is the circular probable error (CPE). The CPE is the radius of a circle for which there is a 50 percent probability that a ballistic missile will impact within the circle.

A simple extension of the CPE concept would be a spherical probable error (SPE), in which three dimensions are considered. Another useful error volume measure, that may be obtained from the covariance matrix of errors, is the equiprobability ellipsoid. This is a surface for which all orientations of the error vector are equally probable. The volume, orientation, and principle axes of this ellipsoid, for respective probabilities, are of interest in space mission studies.

2.0 BASIC STATISTICAL CONCEPTS

Performance errors of a missile system are generally discussed on a probabilistic basis and the concept of a random experiment is fundamental to establishing meaningful measures of performance. Statistical measurements are defined here as those quantities which describe the degree and character of the randomness. The following discussion is intended as a brief review of probability theory as it applies to the statistical measures of vehicle performance; a more complete treatment of the theory may be found in References 1 through 4.

2.1 Probability Distributions:

A random experiment is one whose outcome can be predicted only on a probabilistic basis. It is recalled that if a random experiment is repeated a large number of times, the results (in the aggregate) tend to exhibit consistent properties. For example, consider a specific event A which is sometimes the outcome of an experiment. The ratio of the number of times that A occurs to the number of times the experiment is repeated is called the frequency ratio of the event A. With more and more repetitions of the experiment, this frequency ratio tends toward a constant value P which is the probability that the event A will occur in a single performance of the experiment.

A quantity which expresses the result of a random experiment is called a random variable. If X denotes a random variable, then X will assume different values for each performance of the experiment. A fundamental statistical measure is then the probability $P\ (X = a)$ that X will assume a given value a, or if X is a continuous variable, the probability $P\ (a < X \leq b)$ that X will assume a value greater than a but not larger than b. Alternatively one may speak of cumulative probability, that is the probability $P(X \leq b)$ that X will assume a value no larger than b.

If we know $P\ (X = a)$ for all permissable values of a, we have the discrete probability distribution of the variable X. Similarly, if we know $P\ (a < X \leq b)$ for all values of a and b, we have the continuous probability distribution for the variable X. If we know $P\ (X \leq b)$ for all values of b, we have the distribution function

$$F(x) = P(X \leq x) \tag{2-1}$$

for the variable X.

In dealing with a continuous random variable, the probability distribution function f(x) is defined by considering the interval (x, x + dx). The probability that the variable X will assume a value in the interval (x, x + dx) is written

$$P(x < X \leq x + dx) = f(x)\,dx \tag{2-2}$$

The probability $P(a < X \leq b)$ is then computed from the relationship

$$P(a < X \leq b) = \int_a^b f(x)\,dx \tag{2-3}$$

The distribution function (frequently called the cumulative probability distribution) is then

$$F(x) = \int_0^x f(u)\,du \tag{2-4}$$

A typical probability distribution function and the corresponding cumulative distribution are shown in Figure 2-1 . An obvious property of the probability distribution function is that it is a non-negative number, i.e.,

$$f(x) > 0$$

Obvious properties of the cumulative distribution function are

$$0 \leq F(x) \leq 1$$
$$F(-\infty) = 0$$
$$F(\infty) = 1$$

A physical interpretation can be given $f(x)$ and $F(x)$ by imagining a unit mass, distributed along a straight line such that the fraction of mass concentrated to the left of $X = x$ is equal to $F(x)$. Then the derivative

$$\frac{dF}{dx} = f(x) \tag{2-5}$$

is the density of the unit mass at the point x. $f(x)$ is frequently called the probability density function of the random variable X.

2.2 Joint Probability Distributions

In many cases the result of a random experiment is not expressed by one observed quantity, but by a certain number of simultaneously observed quantities. For example, in the guidance error study, the random experiment involves the simultaneous occurrence of six random variables - the errors in three components of position and three components of velocity. In this event, the one-dimensional concepts are generalized and we speak of the probability that the random variables $X_1, X_2, \ldots, X_n$ will simultaneously belong to the intervals $(a_1\ b_1)$, (a_2, b_2), $\ldots$, (a_n, b_n), respectively. Expressed mathematically

$$P(a_1 < X \leq b_1, \cdots, a_n < X_n \leq b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \cdots x_n)\, dx_1, \cdots, dx_n \tag{2-6}$$

where $f(x_1, x_2, \ldots, x_n)$ is the joint probability distribution function. In general, the individual random variables are not independent of one another. That is the probability that X_i will belong to the interval (a_i, b_i) will depend upon the values assumed by all the other random variables in the joint distribution.

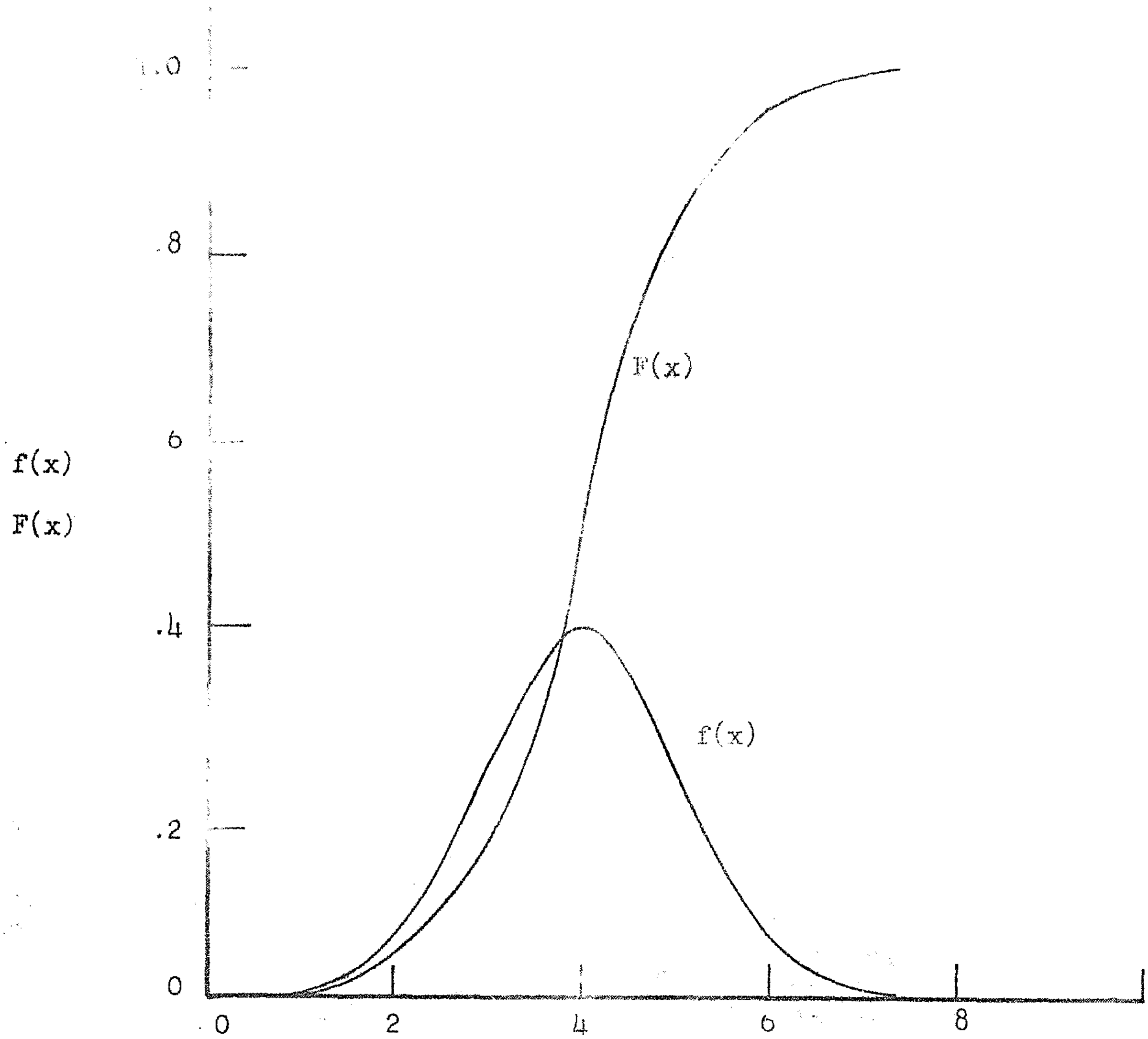


Figure 2-1 - A Typical Probability Distribution Function and the Corresponding Cumulative Distribution

A joint distribution function is defined as follows:

$$F(x_1, x_2, \cdots, x_n) = P(X_1 \leqq x_1, X_2 \leqq x_2, \cdots, X_n \leqq x_n)$$

$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(x_1, \cdots, x_n)\, dx_1, \cdots, dx_n \tag{2-7}$$

Obvious properties of the joint distribution function are

$$F(+\infty, \cdots, +\infty) = 1$$

$$F(-\infty, x_2, \cdots, x_n) = F(x_1, -\infty, \cdots, x_n)$$

$$= F(x_1, x_2, -\infty, \cdots, x_n)$$

$$= \cdots\cdots = 0$$

The probability that $X_i \leq x_i$ without regard to the values assumed by the other random variables is called the marginal distribution of X_i and is defined as

$$F_i(x_i) = \int_{-\infty}^{x_i} du \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, u, \cdots, x_n)\, dx_1, \cdots, dx_n \tag{2-8}$$

where u is a dummy variable for x_i. The marginal frequency function is defined as follows:

$$f_i(x_i) = \frac{dF_i}{dx_i} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n) \prod_{j=1}^{n} dx_j \tag{2-9}$$

where $j \neq i$.

The concept of marginal distributions is useful in guidance error studies where one might be interested in position errors without regard for the velocity errors or vice versa. In this case the respective marginal distribution would involve the joint distribution of only three random variables rather than all six variables. Studies involving impact dispersions would consider only two random variables, downrange miss distance and crossrange miss distance; the third position error would be excluded by the constraint imposed by the earth's surface.

A typical two-dimensional probability distribution function is shown in Figure 2-2 . It is observed that a vertical plane cuts this distribution to form a univariate (conditional) distribution. A conditional distribution $f_{x_i}(x_j)$, $i \neq j$, is defined by the probability that the variable X_j will assume a value within the interval $(x_j, x_j + dx_j)$ on the condition that the variable X_i assumes a specified value x_i. It should also be observed that the intersection of this surface with a horizontal plane yields an ellipse, with the size of the ellipse varying with the height of the cut. These are called equi-probability ellipses.

If for all values a_1, b_1, a_2, b_2, . . ., a_n, b_n, the events

$$
\begin{gathered}
a_1 < X_1 \leq b_1 \\
a_2 < X_2 \leq b_2 \\
\vdots \\
a_n < X_n \leq b_n
\end{gathered}
$$

are independent, then the relationship

$$P(a_1 < X_1 \leq b_1, \cdots, a_n < X_n \leq b_n) = P(a_1 < X \leq b_1) \cdots P(a_n < X \leq b_n) \qquad (2\text{-}10)$$

is valid. Similar relationships for the distribution function and the frequency function follow for independent random variables.

$$
\begin{aligned}
F(x_1, x_2, \cdots, x_n) &= F_1(x_1), F_2(x_2), \cdots, F_n(x_n) \\
f(x_1, x_2, \cdots, x_n) &= f_1(x_1), f_2(x_2), \cdots, f_n(x_n)
\end{aligned} \qquad (2\text{-}11)
$$

2.3 <u>Mean Values</u>

The abscissa of the center of gravity of the probability distribution $f(x)$ constitutes a weighted mean value. This mean value is called the expected value $E(X)$ of the variable X and is defined as follows:

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx \qquad (2\text{-}12)$$

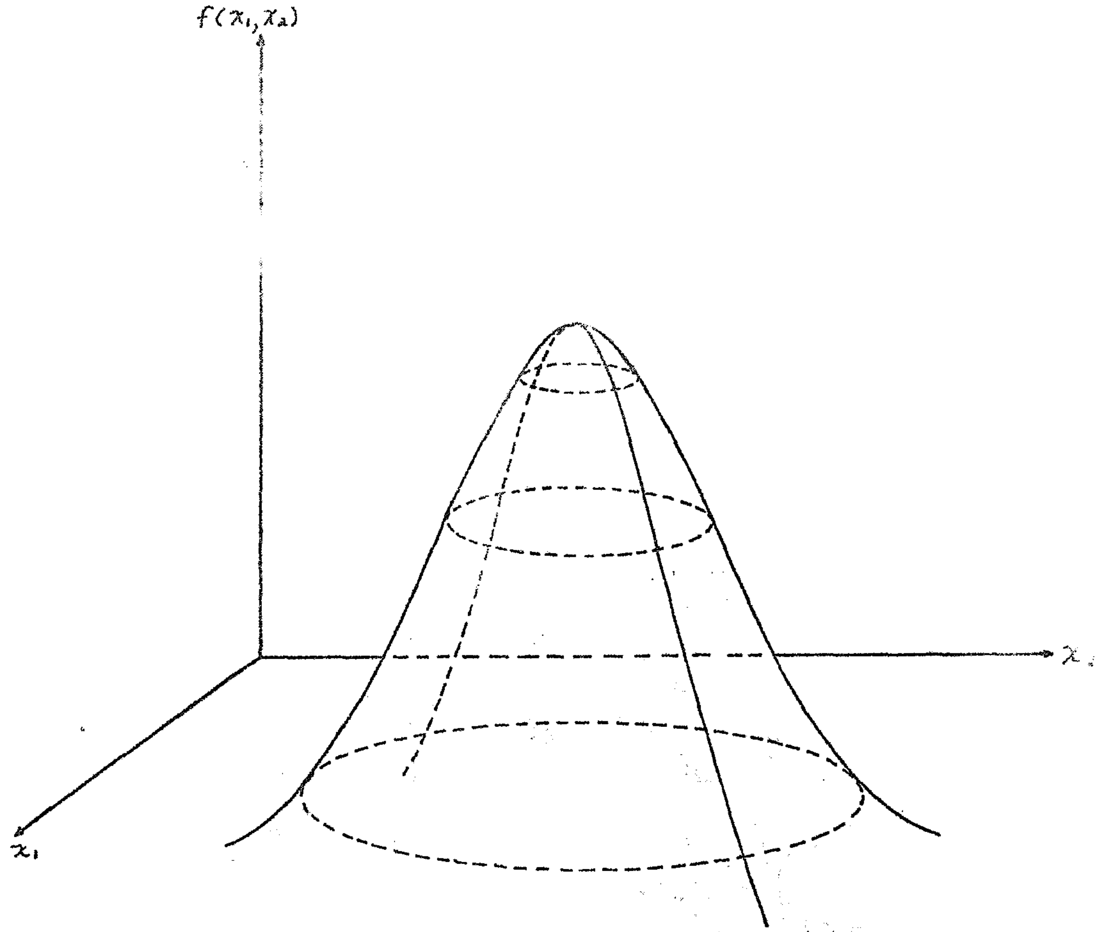


Figure 2-2 - A Typical Two Dimensional Probability Distribution Function

Similarly the expected value of a function $Y = \phi(x)$ is defined as

$$E\left[\phi(x)\right] = \int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx \tag{2-13}$$

The above definition of a mean value is extended directly to joint distributions, which involve the simultaneous occurrence of several random variables. The mean value of the variable X_i is

$$E(X_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_i\, f(x_1, x_2, \cdots x_n)\, dx_1, dx_2 \cdots dx_n \tag{2-14}$$

Consider a linear function, $aX + b$. From Equation (2-13) or (2-14), the mean value is

$$E(aX + b) = aE(X) + b \tag{2-15}$$

Now let us consider the sum, $X_1 + X_2 + \ldots + X_n$ of the random variables in a joint distribution. From Equation (2-14), the mean value is

$$\begin{aligned} E(X_1 + \cdots X_n) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1 + \cdots + x_n)\, f(x_1, \cdots, x_n)\, dx_1, \cdots dx_n \\ &= E(X_1) + E(X_2) + \cdots E(X_n) \end{aligned} \tag{2-16}$$

It should be noted that these addition rules are valid for mean values, whether the random variables are independent or not.

2.4 Covariance and Correlation Coefficients

The mean value of a random variable has been defined as a first order moment of the probability distribution about the origin. This mean value provides a measure of the location of the distribution. Other properties of a distribution may be measured by considering the higher order moments. In particular, the standard deviation or variance and the covariance provide a measure of the dispersion of the random variable about its mean value and the interdependence of the random variables, respectively.

Consider the second order moment about the origin,

$$E(X_i X_j) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} X_i X_j\, f(x_1, \cdots, x_n)\, dx_1, \cdots, dx_n \tag{2-17}$$

If the random variables are independent,

$$E(X_i X_j) = \int_{-\infty}^{\infty} x_i f_i(x_i)\, dx_i \int_{-\infty}^{\infty} x_j f_j(x_j)\, dx_j$$
$$= E(X_i)\, E(X_j) \tag{2-18}$$

where $f_i(x_i)$ and $f_j(x_j)$ are marginal distributions. This multiplication rule is valid only when the random variables are independent.

If the second order moments are computed about the mean values, then the so-called covariance $\sigma_{x_i x_j}$ is obtained, i.e.,

$$\begin{aligned}\sigma_{x_i x_j} &= E\left[(X_i - m_i)(X_j - m_j)\right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - m_i)(x_j - m_j)\, f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n\end{aligned} \tag{2-19}$$

If i = j, this central second order moment is called the variance $\sigma^2_{x_i}$. The standard deviation σ_{x_i} is simply the square root of the variance. The following relationship is established from Equation (2-19)

$$\sigma_{x_i x_j} = E(X_i X_j) - E(X_i)\, E(X_j) \tag{2-20}$$

Correlation coefficients of the variables X_i and X_j are defined as follows:

$$\rho_{x_i x_j} = \sigma_{x_i x_j} \big/ \sigma_{x_i} \sigma_{x_j} \tag{2-21}$$

The correlation coefficients are bounded between -1 and +1 and are equal to 1 for i = j. If $\rho = \pm 1$, there exists a complete linear dependence between the variables X_i and X_j. The values of X_i and X_j will vary in the same sense or the inverse sense when $\rho = 1$ or $\rho = -1$, respectively. Correlation coefficients provide a measure of the degree of linear dependence between the respective variables. If $\rho = 0$, the variables X_i and X_j are completely independent and, as a result, are uncorrelated.

2.5 Covariance Matrix of Errors

The guidance error study is concerned principally with the errors associated with position and velocity measurements in an orthogonal reference frame of cartesian coordinates. The random variables are then the three errors ξ, η, ζ in the scalar components of the position measurement and the three errors $\dot{\xi}, \dot{\eta}, \dot{\zeta}$ in the scalar components of the velocity measurement. The covariance matrix of errors is then a 6 x 6 array of central, second order moments defined by Equation (2-19). This covariance matrix of errors is written

$$C_{66} = \left[\begin{array}{ccc|ccc} \sigma_\xi^2 & \sigma_{\xi\eta} & \sigma_{\xi\zeta} & \sigma_{\xi\dot\xi} & \sigma_{\xi\dot\eta} & \sigma_{\xi\dot\zeta} \\ \sigma_{\eta\xi} & \sigma_\eta^2 & \sigma_{\eta\zeta} & \sigma_{\eta\dot\xi} & \sigma_{\eta\dot\eta} & \sigma_{\eta\dot\zeta} \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & \sigma_\zeta^2 & \sigma_{\zeta\dot\xi} & \sigma_{\zeta\dot\eta} & \sigma_{\zeta\dot\zeta} \\ \hline \sigma_{\dot\xi\xi} & \sigma_{\dot\xi\eta} & \sigma_{\dot\xi\zeta} & \sigma_{\dot\xi}^2 & \sigma_{\dot\xi\dot\eta} & \sigma_{\dot\xi\dot\zeta} \\ \sigma_{\dot\eta\xi} & \sigma_{\dot\eta\eta} & \sigma_{\dot\eta\zeta} & \sigma_{\dot\eta\dot\xi} & \sigma_{\dot\eta}^2 & \sigma_{\dot\eta\dot\zeta} \\ \sigma_{\dot\zeta\xi} & \sigma_{\dot\zeta\eta} & \sigma_{\dot\zeta\zeta} & \sigma_{\dot\zeta\dot\xi} & \sigma_{\dot\zeta\dot\eta} & \sigma_{\dot\zeta}^2 \end{array}\right] \qquad (2\text{-}22)$$

And, by the definition of the covariance, this is a symmetric matrix.

It is observed that the diagonal elements of this covariance matrix are the variances which measure the dispersion of the errors about their mean values, while the off-diagonal elements are the covariances which measure the interdependence of the random variables. Alternately, this covariance matrix may be written in a factored form in terms of the standard deviations and correlation coefficients.

$$C = \left[{}^{\nwarrow}\sigma_{\searrow}\right]\left[\rho\right]\left[{}^{\nwarrow}\sigma_{\searrow}\right] \qquad (2\text{-}22a)$$

where

$$\left[{}^{\nwarrow}\sigma_{\searrow}\right] = \begin{bmatrix} \sigma_\xi & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{\dot\xi} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\dot\eta} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\dot\zeta} \end{bmatrix}$$

and

$$[\rho] = \begin{bmatrix} 1 & \rho_{\xi\eta} & \rho_{\xi\zeta} & \rho_{\xi\dot{\xi}} & \rho_{\xi\dot{\eta}} & \rho_{\xi\dot{\zeta}} \\ \rho_{\eta\xi} & 1 & \rho_{\eta\zeta} & \rho_{\eta\dot{\xi}} & \rho_{\eta\dot{\eta}} & \rho_{\eta\dot{\zeta}} \\ \rho_{\zeta\xi} & \rho_{\zeta\eta} & 1 & \rho_{\zeta\dot{\xi}} & \rho_{\zeta\dot{\eta}} & \rho_{\zeta\dot{\zeta}} \\ \rho_{\dot{\xi}\xi} & \rho_{\dot{\xi}\eta} & \rho_{\dot{\xi}\zeta} & 1 & \rho_{\dot{\xi}\dot{\eta}} & \rho_{\dot{\xi}\dot{\zeta}} \\ \rho_{\dot{\eta}\xi} & \rho_{\dot{\eta}\eta} & \rho_{\dot{\eta}\zeta} & \rho_{\dot{\eta}\dot{\xi}} & 1 & \rho_{\dot{\eta}\dot{\zeta}} \\ \rho_{\dot{\zeta}\xi} & \rho_{\dot{\zeta}\eta} & \rho_{\dot{\zeta}\zeta} & \rho_{\dot{\zeta}\dot{\xi}} & \rho_{\dot{\zeta}\dot{\eta}} & 1 \end{bmatrix}$$

Referring to the definition of the covariance (2-19) and the correlation coefficient (2-21), the matrix of correlation coefficients is seen to be symmetric.

Many guidance error studies may be concerned with position errors without regard for the velocity error or vice versa. In this case, we consider the marginal distribution concept as defined in Section 2.2. By employing the definition of covariance (2-19), and the marginal distribution function (2-9),

$$f_1(\xi,\eta,\zeta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta})\, d\dot{\xi}\, d\dot{\eta}\, d\dot{\zeta} \tag{2-23}$$

or

$$f_2(\dot{\xi},\dot{\eta},\dot{\zeta}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta})\, d\xi\, d\eta\, d\zeta$$

it may be shown that the covariance matrix of errors for the marginal distribution is obtained by simply partitioning the 6 x 6 covariance matrix, Equation (2-22). For the marginal distribution of position errors without regard for velocity errors, the covariance matrix is simply the upper left quadrant of Equation (2-22); for the marginal distribution of velocity errors without regard for the position errors, the covariance matrix is simply the lower right quadrant of Equation (2-22).

2.6 Gaussian Probability Distributions

The Gaussian probability distribution, frequently called the normal probability distribution, is encountered quite often and would generally apply in the problem at hand regarding performance errors due to error sources in the guidance hardware. A major reason for its general applicability is stated by the "Central Limit Theorem". This theorem states that the sum of a large number of random variables is approximately Gaussian distributed regardless of the particular distributions of the individual components of the sum.

The one-dimensional Gaussian probability density function is written

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \tag{2-24}$$

where σ and m denote the standard deviation and mean value respectively of the random variable X. The properties of a Gaussian density function are as follows:

(1) The mode (peak of the probability distribution) calculated by $\frac{df}{dx} = 0$, is located at x = m.

(2) The inflection points, calculated by $\frac{d^2f}{dx^2} = 0$, are located at $x = \pm\sigma$.

The n-dimensional Gaussian probability density function, required in the guidance error study, is written

$$f(\vec{x}_n) = \frac{1}{\sqrt{(2\pi)^n |C_{nn}|}} \exp\left(-\frac{1}{2}\vec{x}_n^T C_{nn}^{-1} \vec{x}_n\right) \tag{2-25}$$

where:

$\vec{x}_n$ is a column vector $(x_1, x_2, \ldots, x_n)^T$ of the errors about their mean values,

$\vec{x}_n^T$ is the row vector,

C_{nn} is the n x n covariance matrix of errors,

$|C_{nn}|$ is the determinant of the covariance matrix.

If we consider the complete 6-dimensional distribution of position and velocity errors,

$$\vec{x}_6 = (\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta})^T$$

and C_{66} is the covariance matrix defined by Equation (2-22). If we are concerned only with the marginal desnity function for position errors without regard for the velocity errors,

$$\vec{x}_3 = (\xi, \eta, \zeta)^T$$

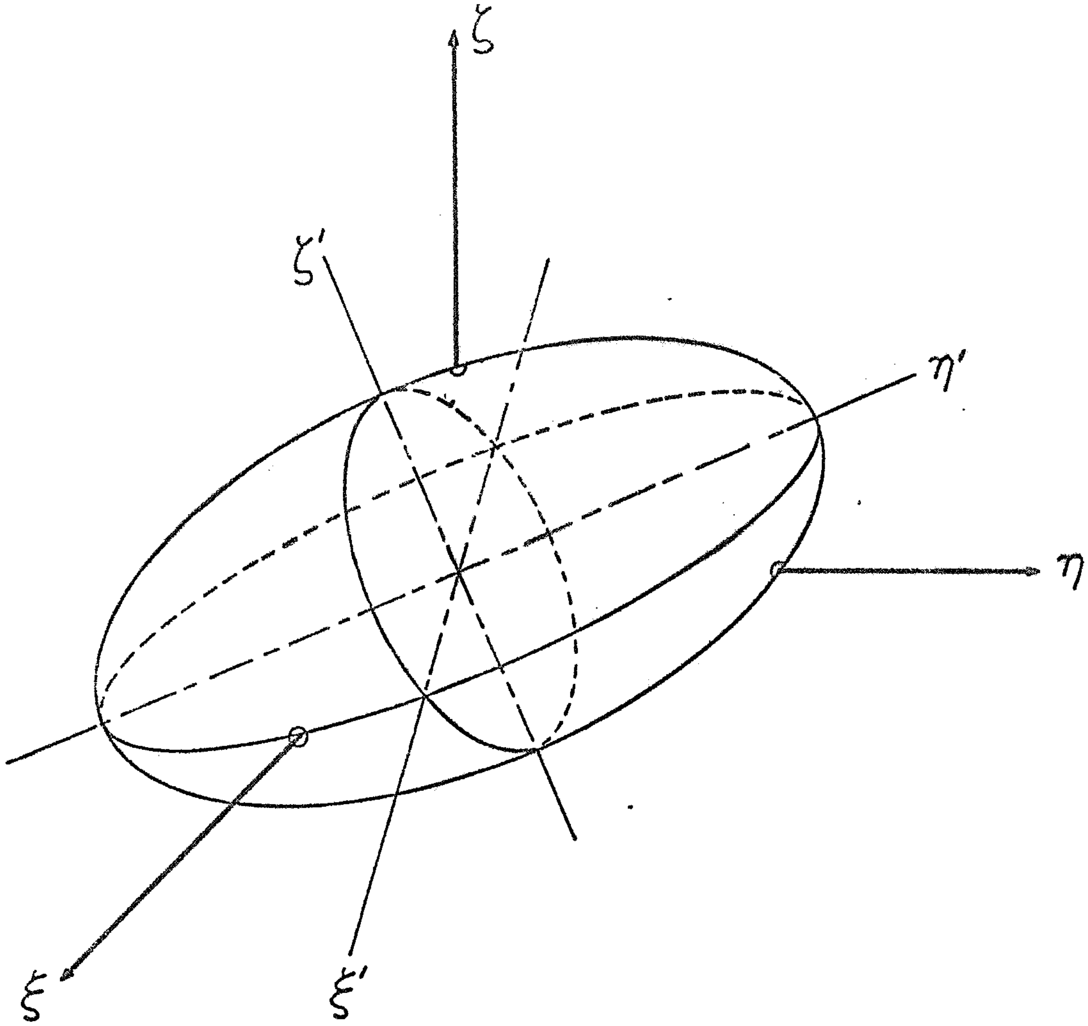
and C_{33} is simply the upper left quadrant of Equation (2-22). The marginal density function for the velocity errors is formed in a similar manner.

This n-dimensional density function describes a family of n-dimensional equiprobability ellipsoids, which are akin to the two-dimensional equiprobability ellipses described in Section 2.2. Each ellipsoid is an equi-probability surface defined by setting the exponential argument of Equation (2-25) to a constant value corresponding to a particular probability, i.e.,

$$k = \bar{x}_n^T \, C_{nn}^{-1} \, \bar{x}_n \tag{2-26}$$

The three-dimensional equiprobability ellipsoids, in particular, lend themselves to useful geometric interpretation. In this regard we would consider the marginal distributions either of the position errors $\bar{x}_3$ or of the velocity errors $\dot{\bar{x}}_3$. Such a surface of equal probability, for the marginal distribution $f_1(\xi, \eta, \zeta)$ is shown in Figure 2-3. It is observed that the origin of the cartesian coordinates (ξ, η, ζ) is coincident with the mean values of the errors. The directions of the principal axes are not, however, aligned with the (ξ, η, ζ) axis system.

The problem of determining the equiprobability ellipsoids and their orientation is treated in subsequent sections. Procedures for obtaining other statistical measurements, such as the error volume associated with the equiprobability ellipsoid and the probability of errors occurring within a given geometrical domain (e.g., a sphere of radius R) are also formulated.



ξ, η, ζ = error measurement axis system

ξ', η', ζ' = principal axes of the equiprobability ellipsoid

Figure 2-3 - A Typical Equiprobability Ellipsoid

3.0 PROPERTIES OF THE COVARIANCE MATRIX

As presented in Section 2.0, the covariance matrix of errors is, in general, an n x n symmetric matrix composed of n variance terms and n(n - 1) covariance terms. The variances, which specify the dispersion of the different errors about their respective means, are on the main diagonal while the covariances, which specify the interdependence of the different errors, occupy all the off-diagonal locations. The useful properties of this matrix arise from its diagonalization - a process in which all the covariance terms are driven to zero and only the variances remain. These properties are:

1. The variance terms remaining are called the eigenvalues of the matrix. These variances correspond to uncorrelated (independent) errors which are associated with measurements in a new frame of reference.

2. Corresponding to each eigenvalue there is an associated eigenvector. Each eigenvector element is a direction cosine and the complete matrix of eigenvectors specifies the orientation of the new axis system with respect to the original frame of reference.

3. Error volumes, corresponding to specific probability levels, are obtainable from the covariance matrix. The volume of the equiprobability ellipsoid is directly proportional to the determinant of the covariance matrix (see page 34). This determinant, the generalized variance, is a frequently used figure of merit and is simply the product of the eigenvalues.

4. The lengths of the semi-axes of the equiprobability ellipsoid are proportional to the eigenvalues; the orientation of the ellipsoidal error volume is given by the eigenvectors (refer to 2 above).

A brief theoretical discussion concerning the calculation of eigenvalues and eigenvectors and diagonalization of the covariance matrix will now be presented followed by two eigenvector computation schemes (algorithms) in the next section.

The eigenvalues of the covariance matrix C are the characteristic or latent roots of the matrix found from the polynomial equation:

$$|C-\lambda I| = \begin{vmatrix} (\sigma_\xi^2-\lambda) & \sigma_{\xi\eta} & \cdots & \sigma_{\xi\zeta} \\ \sigma_{\eta\xi} & (\sigma_\eta^2-\lambda) & \cdots & \sigma_{\eta\zeta} \\ \vdots & \vdots & & \vdots \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & \cdots & (\sigma_\zeta^2-\lambda) \end{vmatrix} = 0 \qquad (3\text{-}1)$$

It should be noted that n λ's, not all of which may be distinct, are found from (3-1). Since the covariance matrix of errors is by its construction always positive definite as well as symmetric, the eigenvalues are all positive real numbers. These eigenvalues also satisfy the matrix equation

$$C\vec{X}_i = \lambda_i \vec{X}_i \qquad i = 1, 2, \cdots n \tag{3-2}$$

where $\vec{X}_i$ is an arbitrary column vector of n elements.

The eigenvectors are the n solution vectors $\vec{\phi}_i$ of (3-2) associated with the n eigenvalues λ_i. Each eigenvector is composed of n elements

$$\{\phi_{i1}, \phi_{i2}, \phi_{i3}, \cdots \phi_{in}\}$$

and satisfies the orthonormal relationship

$$\vec{\phi}_i^{\,T} \vec{\phi}_j = \delta_{ij} \tag{3-3}$$

where δ_{ij} is the Kronecker delta. Thus the magnitude of each eigenvector is unity and the scalar or dot product $\vec{\phi}_i^{\,T} \vec{\phi}_j$ $(i \neq j)$ is zero. This latter result shows that the eigenvectors are linearly independent and mutually orthogonal. Any arbitrary vector in n space can be constructed as a linear combination of the n eigenvectors. In this respect, the eigenvectors are similar to the familiar unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ in three space.

The desired diagonalization of the covariance matrix C can now proceed. We form the eigenvector matrix

$$\Phi = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\phi}_1 & \vec{\phi}_2 & \cdots & \vec{\phi}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{n1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{n2} \\ \phi_{13} & \phi_{23} & \cdots & \phi_{n3} \\ \vdots & \vdots & & \vdots \\ \phi_{1n} & \phi_{2n} & \cdots & \phi_{nn} \end{bmatrix} \tag{3-4}$$

with the eigenvectors as successive columns of Φ. By virtue of Equation (3-3),

$$\Phi^T \Phi = I$$

$$\Phi^{-1} = \Phi^T$$

$$|\Phi| = 1$$

Thus Φ is an orthogonal matrix and performs a transformation of coordinates from one orthogonal set of axes to another. Then, the diagonal covariance matrix $\grave{C}$ is obtained as:

$$\grave{C} = \Phi^T C \Phi = \begin{bmatrix} \sigma_{\xi'}^2 & & & 0 \\ & \sigma_{\eta'}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{\zeta'}^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{3-5}$$

Furthermore, the new coordinates ξ', η', ζ', . . . associated with $\grave{C}$ are related to the old coordinates ξ, η, ζ, . . . associated with C by

$$\begin{aligned} \xi' &= \phi_{11}\xi + \phi_{12}\eta + \phi_{13}\zeta + \cdots\cdots \\ \eta' &= \phi_{21}\xi + \phi_{22}\eta + \phi_{23}\zeta + \cdots\cdots \\ &\vdots \end{aligned} \tag{3-6}$$

and it can be seen that each ϕ_{ij} specifies the component of an old coordinate along a new coordinate axis. Thus, each element of the eigenvector matrix is the direction cosine between an old and new axis. For example

$$\phi_{13} \equiv \cos(\xi', \zeta) \tag{3-7}$$

In this way the eigenvectors exactly specify the orientation of the equiprobability ellipsoid associated with $\grave{C}$.

4.0 EIGENVECTOR ALGORITHMS

The basic statistical quantities of use in the Guidance Error Study are the variances and covariances of position and velocity errors from a reference trajectory. As pointed out in Section 2.5, these quantities are grouped in an array called the covariance matrix of errors. This complete 6 x 6 matrix is written

$$C_{66} = \begin{bmatrix} \sigma_{\xi}^2 & \sigma_{\xi\eta} & \sigma_{\xi\zeta} & \sigma_{\xi\dot{\xi}} & \sigma_{\xi\dot{\eta}} & \sigma_{\xi\dot{\zeta}} \\ \sigma_{\eta\xi} & \sigma_{\eta}^2 & \sigma_{\eta\zeta} & \sigma_{\eta\dot{\xi}} & \sigma_{\eta\dot{\eta}} & \sigma_{\eta\dot{\zeta}} \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & \sigma_{\zeta}^2 & \sigma_{\zeta\dot{\xi}} & \sigma_{\zeta\dot{\eta}} & \sigma_{\zeta\dot{\zeta}} \\ \sigma_{\dot{\xi}\xi} & \sigma_{\dot{\xi}\eta} & \sigma_{\dot{\xi}\zeta} & \sigma_{\dot{\xi}}^2 & \sigma_{\dot{\xi}\dot{\eta}} & \sigma_{\dot{\xi}\dot{\zeta}} \\ \sigma_{\dot{\eta}\xi} & \sigma_{\dot{\eta}\eta} & \sigma_{\dot{\eta}\zeta} & \sigma_{\dot{\eta}\dot{\xi}} & \sigma_{\dot{\eta}}^2 & \sigma_{\dot{\eta}\dot{\zeta}} \\ \sigma_{\dot{\zeta}\xi} & \sigma_{\dot{\zeta}\eta} & \sigma_{\dot{\zeta}\zeta} & \sigma_{\dot{\zeta}\dot{\xi}} & \sigma_{\dot{\zeta}\dot{\eta}} & \sigma_{\dot{\zeta}}^2 \end{bmatrix} \tag{2-22}$$

It is observed that C_{66} contains a total of 36 elements, nine pertaining solely to position, nine pertaining solely to velocity, and the remaining eighteen expressing the correlation between these error components.

If we are concerned only with the errors in position without regard to the errors in velocity, then it was shown in Section 2.5 that C_{66} should be partitioned into four submatrices and only the upper 3 x 3 submatrix of position errors retained.

$$C_{66} = \left[\begin{array}{c|c} C_{33p} & C_{33pv} \\ \hline C_{33vp} & C_{33v} \end{array}\right] \tag{4-1}$$

Similarly, if only the velocity errors are considered, the lower right 3 x 3 submatrix is retained. This procedure was justified in Section 2.5 using the idea of a marginal distribution; thus, the position and velocity submatrices, C_{33p} and C_{33v}, can be calculated from the marginal distribution functions

$f_1(\xi, \eta, \zeta)$ and $f_2(\dot{\xi}, \dot{\eta}, \dot{\zeta})$.

Three dimensional error volumes are associated with either of these submatrices. One particular error volume is the equiprobability ellipsoid discussed in Section 2.6. It is recalled from Section 3.0 that the orientation of this ellipsoidal error volume is given by the eigenvector matrix Φ. A procedure for computing this eigenvector matrix for the position and velocity submatrices will be presented in Section 4.1.

If both the position and velocity errors are important and, especially, if the correlation between position and velocity errors is significant, the entire covariance matrix must be considered rather than the two submatrices separately. Remembering that each element of the eigenvector matrix is a direction cosine, the eigenvector matrix corresponding to C_{66} can be written:

$$\Phi = \begin{bmatrix} \cos(\xi',\xi) & \cos(\eta',\xi) & \cdots & \cos(\dot{\zeta}',\xi) \\ \cos(\xi',\eta) & \cos(\eta',\eta) & \cdots & \cos(\dot{\zeta}',\eta) \\ \cos(\xi',\zeta) & \cos(\eta',\zeta) & \cdots & \cos(\dot{\zeta}',\zeta) \\ & \vdots & & \\ \cos(\xi',\dot{\xi}) & \vdots & \cdots & \cos(\dot{\zeta}',\dot{\xi}) \\ \cos(\xi',\dot{\eta}) & \vdots & \cdots & \cos(\dot{\zeta}',\dot{\eta}) \\ \cos(\xi',\dot{\zeta}) & \vdots & \cdots & \cos(\dot{\zeta}',\dot{\zeta}) \end{bmatrix} \qquad (4\text{-}2)$$

Since only three orthogonal axes can be pictorially represented, Φ is partitioned in the same way as C_{66} in Equation (4-1). In this way a simultaneous set of two three-dimensional coordinate axes are obtained for the position and velocity errors. It is important to note that these coordinate axes are different than those obtained from the submatrices of position and velocity.

The axis system appropriate to the separate covariance submatrices uncorrelate the three position errors or the three velocity errors but do not uncorrelate the position errors from the velocity errors. On the other hand, the 6 x 6 eigenvector matrix calculates six independent (completely uncorrelated) errors in position and velocity. Thus, by taking the upper left quadrant of Φ, a new position error axis system is specified such that the resulting three position errors are all uncorrelated and, furthermore, such that these position errors are independent of the velocity errors. Similarly, the lower right hand quadrant of Equation (4-2) would specify a new velocity error axis system such that the resulting three velocity errors are all uncorrelated from each other and from the position errors.

The usefulness and importance of the 6 x 6 eigenvector matrix can now be appreciated. A hand computational scheme to obtain the eigenvectors appropriate to C_{66} will be presented in Section 4.2.

4.1 Three-Dimensional Submatrices

If the correlation between position and velocity errors is not considered to be significant, the two submatrices contained in the 6 x 6 covariance matrix can be diagonalized separately. In this way the orientation of both the position and velocity equiprobability ellipsoids is obtained such that the position errors are uncorrelated from each other and the velocity errors are uncorrelated from each other. However, the resulting position and velocity errors are still correlated. With this last restriction in mind, the eigenvector calculation scheme will now be presented.

The 3 x 3 submatrix of position errors is written

$$C_{33_p} = \begin{bmatrix} \sigma_\xi^2 & \sigma_{\xi\eta} & \sigma_{\xi\zeta} \\ \sigma_{\eta\xi} & \sigma_\eta^2 & \sigma_{\eta\zeta} \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & \sigma_\zeta^2 \end{bmatrix} \tag{4-3}$$

and the eigenvalues λ_1, λ_2 and λ_3 are considered to be known. Using Equation (3-2) $C_{33p}\vec{\phi}_i = \lambda_i \vec{\phi}_i$.

$$\begin{bmatrix} (\sigma_\xi^2 - \lambda_i) & \sigma_{\xi\eta} & \sigma_{\xi\zeta} \\ \sigma_{\eta\xi} & (\sigma_\eta^2 - \lambda_i) & \sigma_{\eta\zeta} \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & (\sigma_\zeta^2 - \lambda_i) \end{bmatrix} \begin{bmatrix} \phi_{i1} \\ \phi_{i2} \\ \phi_{i3} \end{bmatrix} = 0 \tag{4-4}$$

where i = 1, 2, 3. Then, by solving this set of equations for the eigenvector elements, we obtain

$$\begin{aligned} \frac{\phi_{i1}}{N_i} &= \frac{(\sigma_\eta^2 - \lambda_i)}{\sigma_{\eta\xi}} \left[\frac{\sigma_{\xi\zeta}\sigma_{\eta\xi} - \sigma_{\eta\zeta}(\sigma_\xi^2 - \lambda_i)}{\sigma_{\xi\eta}\sigma_{\eta\xi} - (\sigma_\xi^2 - \lambda_i)(\sigma_\eta^2 - \lambda_i)} \right] - \frac{\sigma_{\eta\zeta}}{\sigma_{\eta\xi}} \\ \frac{\phi_{i2}}{N_i} &= -\left[\frac{\sigma_{\xi\zeta}\sigma_{\eta\xi} - \sigma_{\eta\zeta}(\sigma_\xi^2 - \lambda_i)}{\sigma_{\xi\eta}\sigma_{\eta\xi} - (\sigma_\xi^2 - \lambda_i)(\sigma_\eta^2 - \lambda_i)} \right] \\ \frac{\phi_{i3}}{N_i} &= 1 \end{aligned} \tag{4-5}$$

The norm of each eigenvector is unity; the constant N_i is obtained using the identity

$$\phi_{i1}^2 + \phi_{i2}^2 + \phi_{i3}^2 = 1 \tag{4-6}$$

In each case the positive square root is taken.

Finally, Equations (4-5) are equally valid (with the appropriate changes in subscript notation) for the 3 x 3 submatrix of velocity errors.

4.2 Six-Dimensional Covariance Matrix

If the correlation between the three position errors and the three velocity errors is of importance, then the entire covariance matrix of errors must be diagonalized to determine the 6 x 6 eigenvector matrix. By partitioning this eigenvector matrix, new coordinate axes for position or velocity are obtained for completely uncorrelated errors.

The calculation scheme for the 6 x 6 eigenvector matrix is a direct extension of the formulation in Section 4.1. The six eigenvalues λ_1 through λ_6 are considered known and the equation, analogous to (4-4), is written

$$\begin{bmatrix} (\sigma_\xi^2-\lambda_i) & \sigma_{\xi\eta} & \sigma_{\xi\zeta} & \sigma_{\xi\dot\xi} & \sigma_{\xi\dot\eta} & \sigma_{\xi\dot\zeta} \\ \sigma_{\eta\xi} & (\sigma_\eta^2-\lambda_i) & \sigma_{\eta\zeta} & \sigma_{\eta\dot\xi} & \sigma_{\eta\dot\eta} & \sigma_{\eta\dot\zeta} \\ \sigma_{\zeta\xi} & \sigma_{\zeta\eta} & (\sigma_\zeta^2-\lambda_i) & \sigma_{\zeta\dot\xi} & \sigma_{\zeta\dot\eta} & \sigma_{\zeta\dot\zeta} \\ \sigma_{\dot\xi\xi} & \sigma_{\dot\xi\eta} & \sigma_{\dot\xi\zeta} & (\sigma_{\dot\xi}^2-\lambda_i) & \sigma_{\dot\xi\dot\eta} & \sigma_{\dot\xi\dot\zeta} \\ \sigma_{\dot\eta\xi} & \sigma_{\dot\eta\eta} & \sigma_{\dot\eta\zeta} & \sigma_{\dot\eta\dot\xi} & (\sigma_{\dot\eta}^2-\lambda_i) & \sigma_{\dot\eta\dot\zeta} \\ \sigma_{\dot\zeta\xi} & \sigma_{\dot\zeta\eta} & \sigma_{\dot\zeta\zeta} & \sigma_{\dot\zeta\dot\xi} & \sigma_{\dot\zeta\dot\eta} & (\sigma_{\dot\zeta}^2-\lambda_i) \end{bmatrix} \begin{bmatrix} \phi_{i1} \\ \phi_{i2} \\ \phi_{i3} \\ \phi_{i4} \\ \phi_{i5} \\ \phi_{i6} \end{bmatrix} = 0 \qquad (4\text{-}7)$$

where i = 1, 2, 3, . . . 6.

The scheme is obtained by solving this set of six equations in six unknowns. The scheme is long and tedious and the computations are performed in the following steps:

STEP 1:

$$\sum_{ij} = (\sigma_j^2 - \lambda_i)$$

where: i denotes the eigenvector and associated eigenvalue

i = 1, 2, . . . 6

j denotes the term

$j = 1 \longrightarrow \xi$ $\qquad j = 4 \longrightarrow \dot\xi$

$j = 2 \longrightarrow \eta$ $\qquad j = 5 \longrightarrow \dot\eta$

$j = 3 \longrightarrow \zeta$ $\qquad j = 6 \longrightarrow \dot\zeta$

STEP 2:

$$\alpha_i = (\sigma_{\xi\eta}\sigma_{\eta\xi} - \Sigma_{i1}\Sigma_{i2})$$

$$\beta_i = (\sigma_{\xi\eta}\sigma_{\zeta\xi} - \Sigma_{i1}\sigma_{\zeta\eta})$$

$$\gamma_i = (\sigma_{\xi\eta}\sigma_{\dot{\xi}\xi} - \Sigma_{i1}\sigma_{\dot{\xi}\eta})$$

$$\delta_i = (\sigma_{\xi\eta}\sigma_{\dot{\eta}\xi} - \Sigma_{i1}\sigma_{\dot{\eta}\eta})$$

$$\epsilon_i = (\sigma_{\xi\dot{\zeta}}\sigma_{\eta\xi} - \Sigma_{i1}\sigma_{\eta\dot{\xi}})$$

$$\zeta_i = (\sigma_{\xi\zeta}\sigma_{\eta\xi} - \Sigma_{i1}\sigma_{\eta\zeta})$$

$$\eta_i = (\sigma_{\xi\dot{\eta}}\sigma_{\eta\xi} - \Sigma_{i1}\sigma_{\eta\dot{\eta}})$$

$$\theta_i = (\sigma_{\xi\dot{\xi}}\sigma_{\eta\xi} - \Sigma_{i1}\sigma_{\eta\dot{\xi}})$$

STEP 3:

$$A_{i1} = \epsilon_i\beta_i - (\sigma_{\xi\dot{\zeta}}\sigma_{\zeta\xi} - \Sigma_{i1}\sigma_{\zeta\dot{\xi}})\alpha_i$$

$$A_{i2} = \zeta_i\gamma_i - (\sigma_{\xi\zeta}\sigma_{\dot{\xi}\xi} - \Sigma_{i1}\sigma_{\dot{\xi}\zeta})\alpha_i$$

$$A_{i3} = \zeta_i\beta_i - (\sigma_{\xi\zeta}\sigma_{\zeta\xi} - \Sigma_{i1}\Sigma_{i3})\alpha_i$$

$$A_{i4} = \epsilon_i\gamma_i - (\sigma_{\xi\dot{\zeta}}\sigma_{\dot{\xi}\xi} - \Sigma_{i1}\sigma_{\dot{\xi}\dot{\zeta}})\alpha_i$$

$$A_{i5} = \theta_i\beta_i + (\sigma_{\xi\dot{\xi}}\sigma_{\zeta\xi} - \Sigma_{i1}\sigma_{\zeta\dot{\xi}})\alpha_i$$

$$A_{i6} = \theta_i\gamma_i - (\sigma_{\xi\dot{\xi}}\sigma_{\dot{\xi}\xi} - \Sigma_{i1}\Sigma_{i4})\alpha_i$$

$$A_{i7} = \zeta_i\delta_i - (\sigma_{\xi\zeta}\sigma_{\dot{\eta}\xi} - \Sigma_{i1}\sigma_{\dot{\eta}\zeta})\alpha_i$$

$$A_{i8} = \epsilon_i\delta_i - (\sigma_{\xi\dot{\zeta}}\sigma_{\dot{\eta}\xi} - \Sigma_{i1}\sigma_{\dot{\eta}\dot{\zeta}})\alpha_i$$

$$A_{i9} = \theta_i\delta_i - (\sigma_{\xi\dot{\xi}}\sigma_{\dot{\eta}\xi} - \Sigma_{i1}\sigma_{\dot{\eta}\dot{\xi}})\alpha_i$$

$$A_{i10} = \eta_i\beta_i - (\sigma_{\xi\dot{\eta}}\sigma_{\zeta\xi} - \Sigma_{i1}\sigma_{\zeta\dot{\eta}})\alpha_i$$

$$A_{i11} = \eta_i\gamma_i - (\sigma_{\xi\dot{\eta}}\sigma_{\dot{\xi}\xi} - \Sigma_{i1}\sigma_{\dot{\xi}\dot{\eta}})\alpha_i$$

$$A_{i12} = \eta_i\delta_i - (\sigma_{\xi\dot{\eta}}\sigma_{\dot{\eta}\xi} - \Sigma_{i1}\Sigma_{i5})\alpha_i$$

STEP 4: $\phi_{i6} = C_i \qquad i = 1,2,\cdots,6$

$$B_{i1} = \frac{A_{i1}\,A_{i2} - A_{i3}\,A_{i4}}{A_{i5}\,A_{i2} - A_{i3}\,A_{i6}}$$

$$B_{i2} = \frac{A_{i10}\,A_{i2} - A_{i3}\,A_{i11}}{A_{i5}\,A_{i2} - A_{i3}\,A_{i6}}$$

$$B_{i3} = \frac{A_{i10}\,A_{i7} - A_{i3}\,A_{i12}}{A_{i5}\,A_{i7} - A_{i3}\,A_{i9}}$$

$$B_{i4} = \frac{A_{i1}\,A_{i7} - A_{i3}\,A_{i8}}{A_{i5}\,A_{i7} - A_{i3}\,A_{i9}}$$

$$B_{i5} = \frac{A_{i10}}{A_{i3}} - \frac{A_{i5}}{A_{i3}}B_{i2}$$

$$B_{i6} = \frac{A_{i1}}{A_{i3}} - \frac{A_{i5}}{A_{i3}}B_{i1}$$

STEP 5: $$\phi_{i5} = -C_i\left[\frac{B_{i1} - B_{i4}}{B_{i2} - B_{i3}}\right] = -C_i\,G_i$$

$$\phi_{i4} = -C_i\left[B_{i1} - B_{i2}G_i\right] = -C_i\,H_i$$

$$\phi_{i3} = -C_i\left[B_{i6} - B_{i5}G_i\right] = -C_i\,J_i$$

STEP 6:

$$B_{i7} = \left\{\frac{\eta_i}{\alpha_i} - \frac{A_{i10}\beta_i}{A_{i3}\alpha_i}\right\} - \left\{\frac{\theta_i}{\alpha_i} - \frac{A_{i5}\beta_i}{A_{i3}\alpha_i}\right\}B_{i2}$$

$$B_{i8} = \left\{\frac{\epsilon_i}{\alpha_i} - \frac{A_{i1}\beta_i}{A_{i3}\alpha_i}\right\} - \left\{\frac{\theta_i}{\alpha_i} - \frac{A_{i5}\beta_i}{A_{i3}\alpha_i}\right\}B_{i1}$$

yielding $$\phi_{i2} = -C_i\left[B_{i8} - B_{i7}G_i\right] = -C_i\,K_i$$

STEP 7:

$$d_{i1} = \sigma_{\xi\eta} / \Sigma_{i1} \qquad d_{i4} = \sigma_{\xi\dot{\eta}} / \Sigma_{i1}$$

$$d_{i2} = \sigma_{\xi\zeta} / \Sigma_{i1} \qquad d_{i5} = \sigma_{\xi\dot{\zeta}} / \Sigma_{i1}$$

$$d_{i3} = \sigma_{\xi\dot{\xi}} / \Sigma_{i1}$$

STEP 8:

$$f_{i1} = d_{i2} - d_{i1}\frac{\beta_i}{\alpha_i} \qquad f_{i3} = d_{i4} - d_{i1}\frac{\eta_i}{\alpha_i}$$

$$f_{i2} = d_{i3} - d_{i1}\frac{\theta_i}{\alpha_i} \qquad f_{i4} = d_{i5} - d_{i1}\frac{\epsilon_i}{\alpha_i}$$

STEP 9:

$$B_{i9} = \left\{f_{i3} - f_{i1}\frac{A_{i10}}{A_{i3}}\right\} - \left\{f_{i2} - f_{i1}\frac{A_{i5}}{A_{i3}}\right\}B_{i2}$$

$$B_{i10} = \left\{f_{i4} - f_{i1}\frac{A_{i1}}{A_{i3}}\right\} - \left\{f_{i2} - f_{i1}\frac{A_{i5}}{A_{i3}}\right\}B_{i1}$$

yielding

$$\phi_{i1} = -C_i\left(B_{i10} - B_{i9}G_i\right) = -C_i L_i$$

STEP 10:

Now, G_i, H_i, J_i, K_i, and L_i are known numbers; we solve for C_i as follows

$$\phi_{i1}^2 + \phi_{i2}^2 + \phi_{i3}^2 + \phi_{i4}^2 + \phi_{i5}^2 + \phi_{i6}^2 =$$

$$C_i^2\left(L_i^2 + K_i^2 + J_i^2 + H_i^2 + G_i^2 + 1\right) = 1$$

The positive square root is taken in each case. Thus with a total of 54 calculations per eigenvector, the six eigenvectors can be calculated.

As a final note, both the three and six-dimensional schemes have been verified using the digital computer.

5.0 EQUIPROBABILITY ELLIPSOIDS

In this statistical treatment of performance errors due to guidance hardware errors, the Gaussian distribution of these performance errors is a fundamental assumption. It has been pointed out in Section 2.6 that a family of equiprobability ellipsoids can be defined from the n-dimensional Gaussian probability density function by simply setting the exponential argument of this function to a constant k. This constant is termed the equiprobability parameter. If the integration of this probability density over the n-dimensional ellipsoidal volume can be accomplished analytically, the total probability P_n can be uniquely related to the parameter k. In the following section, the steps involved in this integration will be presented and two closed form expressions relating P_n to k (valid for n even or odd) will be derived. In addition, Section 5.2 will show how the lengths of the ellipsoidal semi-axes are related to k and Section 5.3 will present two simple equations for the volumes of equiprobability ellipsoids of either even or odd dimension.

5.1 The Equiprobability Parameter

The n-dimensional Gaussian probability density function (see Section 2.6) for the n performance errors $x_1, x_2, x_3, \ldots x_n$ is written

$$f(\vec{x}_n) = \frac{1}{\sqrt{(2\pi)^n |C_{nn}|}} \exp\left(-\tfrac{1}{2} \vec{x}_n^{\,T} C_{nn}^{-1} \vec{x}_n\right) \qquad (2\text{-}25)$$

The task at hand is to integrate this expression over the n-dimensional ellipsoidal volume.

First, we recognize that $\vec{x}_n^{\,T} C_{nn}^{-1} \vec{x}_n$ is simply a quadratic form. If this quadratic form can be made canonical, then the integration can proceed. By using the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and eigenvectors $\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_n$ of the covariance matrix C_{nn}, an orthogonal transformation of coordinates is performed yielding

$$\vec{y}_n = \Phi^{-1} \vec{x}_n = \Phi^T \vec{x}_n \qquad (5\text{-}1)$$

and

$$\tilde{C}_{nn} = \Phi^T C_{nn} \Phi = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_n \end{bmatrix} \qquad (5\text{-}2)$$

$$\therefore \quad \tilde{C}_{nn}^{-1} = \begin{bmatrix} 1/\lambda_1 & & 0 \\ & 1/\lambda_2 & \\ & & \ddots \\ 0 & & 1/\lambda_n \end{bmatrix} \qquad (5\text{-}3)$$

Now, equating probability elements,

$$f(\vec{x}_n)\, dx_1\, dx_2 \cdots dx_n = f(\vec{y}_n)\, |J|\, dy_1\, dy_2 \cdots dy_n \qquad (5\text{-}4)$$

where $|J|$ is the Jacobian determinant defined as

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \qquad (5\text{-}5)$$

But $\vec{x_n} = \Phi \vec{y_n}$ and, by using Equation (3-4), we obtain

$$|J| = \begin{vmatrix} \phi_{11} & \phi_{21} & \cdots & \phi_{n1} \\ \phi_{12} & \phi_{22} & \cdots & \phi_{n2} \\ \vdots & \vdots & & \vdots \\ \phi_{1n} & \phi_{2n} & \cdots & \phi_{nn} \end{vmatrix} = 1 \qquad (5\text{-}6)$$

Also,

$$C_{nn} = \Phi \grave{C}_{nn} \Phi^T$$

$$C_{nn}^{-1} = (\Phi \grave{C}_{nn} \Phi^T)^{-1} = \Phi \grave{C}_{nn}^{-1} \Phi^T \qquad (5\text{-}7)$$

$$\therefore \quad \vec{x_n}^T C_{nn}^{-1} \vec{x_n} = (\Phi \vec{y_n})^T \Phi \grave{C}_{nn}^{-1} \Phi^T \Phi \vec{y_n}$$

$$= \vec{y_n} \grave{C}_{nn}^{-1} \vec{y_n}$$

Now, using (5-3), the quadratic form can be written:

$$\vec{y_n}^T \grave{C}_{nn} \vec{y_n} = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} + \cdots \frac{y_n^2}{\lambda_n} \qquad (5\text{-}8)$$

Referring to Equation (2-26), the definition of equiprobability ellipsoids is simply

$$\vec{x_n}^T C_{nn}^{-1} \vec{x_n} = k = \vec{y_n}^T \grave{C}_{nn}^{-1} \vec{y_n} \qquad (5\text{-}9)$$

Using Equations (5-6), (5-7), and (5-8), Equation (5-4) becomes

$$f(\bar{y}_n)\, dy_1\, dy_2 \cdots dy_n =$$

$$\frac{1}{\sqrt{(2\pi)^n |C_{nn}|}} \exp\left(-\frac{1}{2}\left(\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} + \cdots + \frac{y_n^2}{\lambda_n}\right)\right) dy_1\, dy_2 \cdots dy_n \qquad (5\text{-}10)$$

One further basic simplification can be made:

$$|C_{nn}| = |\Phi \grave{C}_{nn} \Phi^T| = |\Phi||\grave{C}_{nn}||\Phi^T| = |\grave{C}_{nn}|$$

and by (5-2)

$$|\grave{C}_{nn}| = \lambda_1 \lambda_2 \cdots \lambda_n$$

Therefore, the final canonical form to integrate is

$$f(\bar{y}_n)\, dy_1 \cdots dy_n = \frac{1}{\sqrt{(2\pi)^n \lambda_1 \cdots \lambda_n}}\, e^{-\frac{1}{2}\left(\frac{y_1^2}{\lambda_1} + \cdots + \frac{y_n^2}{\lambda_n}\right)}\, dy_1 \cdots dy_n \qquad (5\text{-}11)$$

Now we seek to integrate this subject to the constraint $\frac{y_1^2}{K\lambda_1} + \cdots + \frac{y_n^2}{K\lambda_n} = 1$; this constraint is simply an n-dimensional ellipsoid with semi-axes $\sqrt{K\lambda_1}, \sqrt{K\lambda_2}, \cdots, \sqrt{K\lambda_n}$ (refer to Section 5.2). In order to eliminate the dependence on a specific covariance matrix ($\lambda_1, \lambda_2, \cdots, \lambda_n$), the coordinates are again transformed such that the n-dimensional ellipsoid becomes an n-dimensional sphere of radius 1.

$$\frac{y_1^2}{K\lambda_1} + \frac{y_2^2}{K\lambda_2} + \cdots + \frac{y_n^2}{K\lambda_n} = z_1^2 + z_2^2 + \cdots + z_n^2 = 1 \qquad (5\text{-}12)$$

Then

$$f(\bar{z})\,|J_z|\, dz_1\, dz_2 \cdots dz_n = \frac{1}{\sqrt{(2\pi)^n \lambda_1 \lambda_2 \cdots \lambda_n}}\, e^{-\frac{K}{2}(z_1^2 + z_2^2 + \cdots + z_n^2)} \times$$

$$\sqrt{K\lambda_1}\, dz_1\, \sqrt{K\lambda_2}\, dz_2 \cdots \sqrt{K\lambda_n}\, dz_n \qquad (5\text{-}13)$$

$$= \frac{K^{n/2}}{(2\pi)^{n/2}}\, e^{-\frac{K}{2}(z_1^2 + z_2^2 + \cdots + z_n^2)}\, dz_1\, dz_2 \cdots dz_n \qquad (5\text{-}14)$$

where

$$|J_z| = k^{n/2}\sqrt{\lambda_1 \lambda_2 \cdots \lambda_n}$$

Since the rectangular coordinates z_i are not independent by virtue of Equation (5-12), the n-dimensional spherical coordinates r, θ_1, θ_2,. . . θ_{n-1} are introduced. These are defined as

$$\begin{aligned} Z_1 &= r\cos\Theta_1 \\ Z_2 &= r\sin\Theta_1\cos\Theta_2 \\ Z_3 &= r\sin\Theta_1\sin\Theta_2\cos\Theta_3 \\ &\vdots \\ Z_n &= r\sin\Theta_1\sin\Theta_2\cdots\sin\Theta_{n-1} \end{aligned} \tag{5-15}$$

Jacobian in this case is

$$|J_{r,\Theta_i}| = r^{n-1}\sin^{n-2}\Theta_1 \sin^{n-3}\Theta_2 \cdots \sin\Theta_{n-2} \tag{5-16}$$

and the appropriate limits are

$$\begin{aligned} &(0 < r \leq 1) \\ &(0 < \theta_1,\cdots,\theta_{n-2} \leq \pi) \\ &(0 < \theta_{n-1} \leq 2\pi) \end{aligned} \tag{5-17}$$

Thus, the total probability is

$$\begin{aligned} P_n(k) = \frac{k^{n/2}}{(2\pi)^{n/2}} \int_0^1 \int_0^{\pi} \cdots \int_0^{2\pi} e^{-\frac{kr^2}{2}\left[\cos^2\theta_1 + \sin^2\theta_1(\cos^2\theta_2 \right.} & \\ \left. + \sin^2\theta_2(\cos^2\theta_3 + \cdots\cdots + \sin^2\theta_{n-1}))\cdots)\right]} & \\ \times\, r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, dr\, d\theta_1 d\theta_2 \cdots d\theta_{n-1} & \\ n > 1 & \end{aligned} \tag{5-18}$$

which reduces to the n independent integrals

$$P_n(K) = \frac{K^{n/2}}{(2\pi)^{n/2}} \int_0^1 r^{n-1} e^{-\frac{Kr^2}{2}} dr \int_0^{\pi} \sin^{n-2}\theta_1 d\theta_1$$

$$\times \int_0^{\pi} \sin^{n-3}\theta_2 d\theta_2 \cdots \int_0^{\pi} \sin\theta_{n-2} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \qquad n > \tag{5-19}$$

For n = 2, 3, and 6, this formula yields

$$P_2(K) = \frac{K}{2\pi} \int_0^1 r e^{-\frac{Kr^2}{2}} dr \int_0^{2\pi} d\theta_1 = 1 - e^{-K/2} \tag{5-20}$$

$$P_3(K) = \sqrt{\frac{2}{\pi}} \int_0^1 r^2 e^{-\frac{Kr^2}{2}} dr$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{\sqrt{K}} e^{-\beta^2/2} d\beta - \sqrt{K}\, e^{-K/2} \right] \tag{5-21}$$

But the integral can be cast into the form of the Error Function erf(x) = $= \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ Then

$$P_3(K) = \operatorname{erf}\left(\sqrt{\frac{K}{2}}\right) - \sqrt{\frac{2K}{\pi}}\, e^{-K/2}$$

$$P_6(K) = 1 - e^{-K/2}\left(1 + \frac{1}{2}K + \frac{1}{8}K^2\right) \tag{5-22}$$

Finally, by mathematical induction, Equation (5-19) was reduced to two general formulae, valid for any even or odd n greater than 1.

$$P_n(K) = 1 - e^{-K/2} \sum_{j=0}^{\left(\frac{n}{2}-1\right)} \frac{1}{2^j\, j!} K^j; \qquad n = 2, 4, 6 \tag{5-23}$$

$$P_n(K) = \operatorname{erf}\left(\sqrt{\frac{K}{2}}\right) - \sqrt{\frac{2}{\pi}}\, e^{-K/2} \sum_{j=0}^{\frac{(n-3)}{2}} \frac{2^j\, j!}{(2j+1)!} K^{j+1/2} \tag{5-24}$$

$$n = 3, 5, 7, \cdots$$

In summary, two general formulae, Equations (5-23) and (5-24) have been derived which relate the total probability P_n to the equiprobability parameter k. It should be noted that these expressions are independent of the covariance matrix C_{nn} and, hence, valid for any covariance matrix.

However, the usual case of interest is to pick k for a given value of P_n. Since the forms of Equations (5-23) and (5-24) do not permit the analytic determination of k as a function of P_n, Figure (5.1) is presented. This figure is a plot of $P_1(k)$ through $P_6(k)$ versus k; the value of k associated with any value of P_n can be read directly from the figure. It should be noted that the familiar 1, 2, and 3 "sigma" levels are given by k equal to 1, 4, and 9 respectively; i.e., a square root relationship holds. In addition, Table 5.1 gives the values of k associated with a wide range of probability levels for the three-dimensional case.

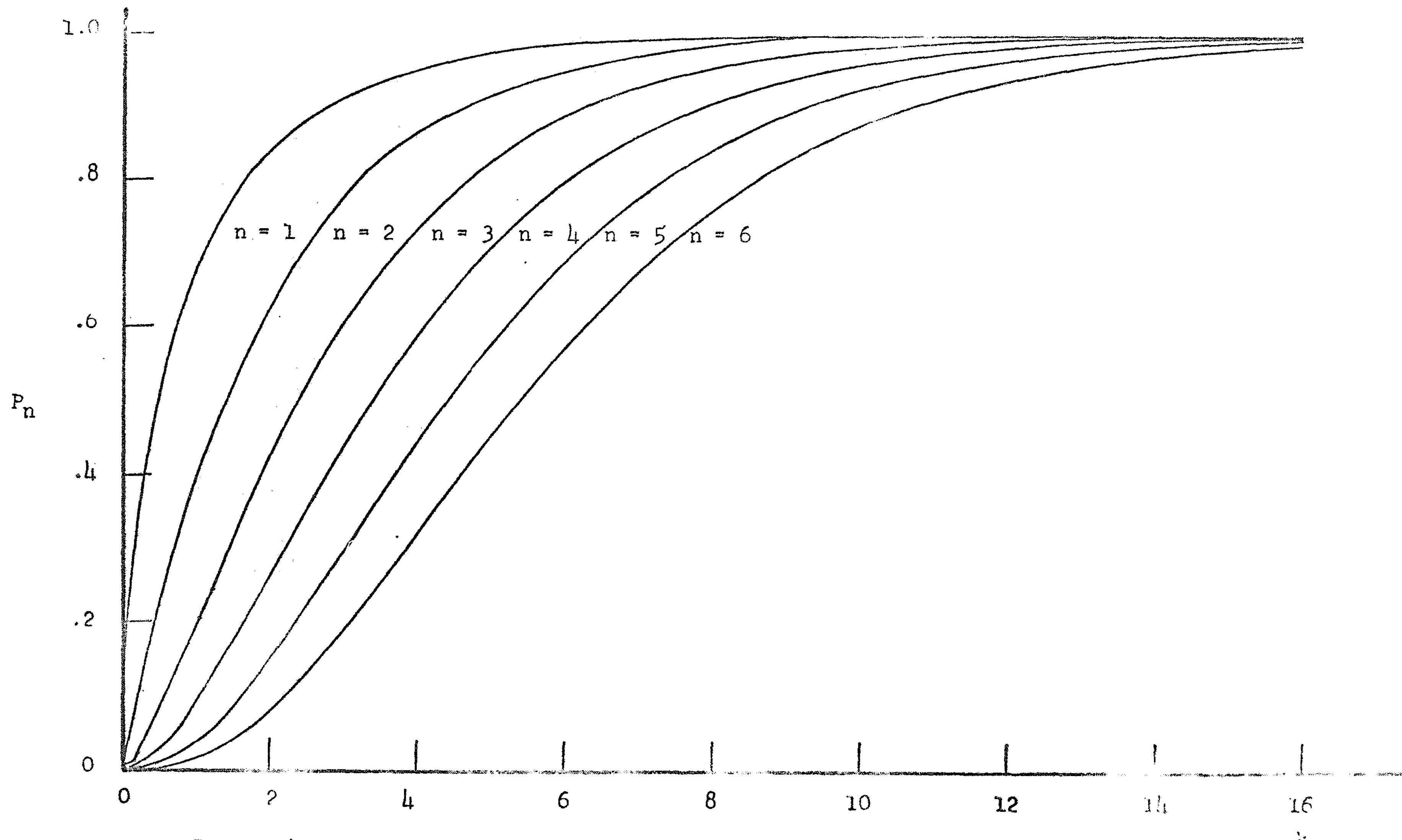


Figure 5.1 - Cumulative Probability as a Function of Equiprobability Parameter

TABLE 5.1

PROBABILITY LEVEL AND EQUIPROBABILITY PARAMETER k

P_3 (percent)	k
10	0.584
19.875 (1σ)	1.000
20	1.004
25	1.210
30	1.419
40	1.867
50	2.365
60	2.943
70	3.675
73.854 (2σ)	4.000
75	4.109
80	4.652
85	5.326
90	6.250
95	7.812
97	8.967
97.071 (3σ)	9.000
98	9.910
99.5	12.870

5.2 Lengths of Ellipsoidal Principal Axes

Equiprobability ellipsoids are defined from the Gaussian probability density function through the use of the equiprobability parameter k. The exponential argument $\vec{x}_n^{\,T} C_{nn}^{-1} \vec{x}_n$ is simply set equal to k. Referring to Equations (5-8) and (5-9), we have

$$\vec{x}_n^{\,T} C_{nn}^{-1} \vec{x}_n = \vec{y}_n^{\,T} \tilde{C}_{nn}^{-1} \vec{y}_n = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} + \cdots + \frac{y_n^2}{\lambda_n} = k \tag{5-25}$$

The general form for an ellipse of 3 dimensions is

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} = 1 \tag{5-26}$$

where a, b, and c are the lengths of the principal axes. In n dimensions, this form expands to:

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} + \frac{y_3^2}{c^2} + \frac{y_4^2}{d^2} + \frac{y_5^2}{e^2} + \cdots = 1 \tag{5-27}$$

Comparing (5-25) and (5-27), we obtain

$$a = \sqrt{k\lambda_1}\ ; \quad b = \sqrt{k\lambda_2}\ ; \quad c = \sqrt{k\lambda_3}\ ; \ \cdots \tag{5-28}$$

for the lengths of the principal axes of the equiprobability ellipsoid.

5.3 Error Volumes

The two unique relationships between the total probability P_n and the equiprobability parameter k (for n even or odd) were derived in Section 5.1. Corresponding to each equiprobability ellipsoid specified by k, there is an associated ellipsoidal error volume. Two general equations for these error volumes, (applicable for n even or odd), will now be derived.

In general, the n-dimensional error volume is:

$$V_n = \int \int \cdots \int dy_1\, dy_2\, dy_3 \cdots dy_n \tag{5-29}$$

where the integration is to be carried out over the ellipsoid specified by:

$$\frac{y_1^2}{k\lambda_1} + \frac{y_2^2}{k\lambda_2} + \cdots + \frac{y_n^2}{k\lambda_n} = 1$$

Following Section 5.1, the ellipsoid is transformed into an n-dimensional sphere of radius 1 by Equation (5-12). Then:

$$V_n = K^{n/2}\sqrt{|C_{nn}|}\int\int\cdots\int dz_1 dz_2 dz_3 \cdots dz_n \qquad (5\text{-}30)$$

Here we have used the fact that $|C_{nn}|$, the so-called generalized variance, is simply the product of the eigenvalues.

Next the n-dimensional spherical coordinates given by (5-15) are incorporated. We obtain

$$V_n = K^{n/2}\sqrt{|C_{nn}|}\int_0^1\int_0^{\pi}\cdots\int_0^{2\pi} r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, dr\, d\theta_1 d\theta_2 \cdots d\theta_{n-2}\, d\theta_{n-1} \qquad (5\text{-}31)$$

This obviously reduces to n independent integrals and, for n even, the general form can be shown to be:

$$V_n = \frac{(\pi K)^{n/2}\sqrt{|C_{nn}|}}{(n/2)!}\ ; \quad n = 2, 4, 6, \cdots \qquad (5\text{-}32)$$

If n is odd, $\frac{n}{2}$ is a fraction and $(\frac{n}{2})!$ must be written in a more appropriate form. We incorporate the Gamma function defined as follows

$$\Gamma(m) = (m-1)!$$

then $\Gamma(m+1) = (m)!$

setting $m = \frac{n}{2}$, it can be shown that

$$\left(\frac{n}{2}\right)! = \Gamma\left(\frac{n}{2}+1\right) = \frac{1\cdot 3\cdot 5\cdot 7\cdots n}{2^{\frac{n+1}{2}}}\sqrt{\pi} \qquad (5\text{-}33)$$

Also

$$1\cdot 3\cdot 5\cdot 7\cdots n = \frac{n!}{2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!} \qquad (5\text{-}34)$$

Thus, using (5-33) and (5-34), the ellipsoidal volume for n odd is

$$V_n = \frac{2^n\left(\frac{n-1}{2}\right)!\ \pi^{\frac{n-1}{2}}\ K^{n/2}\sqrt{|C_{nn}|}}{n!}\ ; \quad n = 3, 5, 7, \cdots \qquad (5\text{-}35)$$

As examples,

$$V_2 = \pi K \sqrt{|C_{22}|}$$

$$V_3 = \frac{4}{3} \pi K^{3/2} \sqrt{|C_{33}|}$$

$$V_6 = \frac{1}{6} \pi^3 K^3 \sqrt{|C_{66}|}$$

It should be noted that in all cases the ellipsoidal volume is directly proportional to the square root of the generalized variance.

Finally, for preliminary estimates, it is shown in Section 7.0 that a conservative estimate for the error volume is obtained by simply neglecting all the covariance terms in C_{nn}. This estimate is conservative since the resulting generalized variance is larger than the actual quantity; hence, for a specified probability, the associated error volume is larger.

For example, referring to Section 2.5, the generalized variance can be written in 3 dimensions as:

$$|C_{33}| = \sigma_\xi^2 \sigma_\eta^2 \sigma_\zeta^2 \left(1 + 2\rho_{\xi\eta} \rho_{\xi\zeta} \rho_{\eta\zeta} - \rho_{\eta\zeta}^2 - \rho_{\xi\eta}^2 - \rho_{\xi\zeta}^2 \right) \tag{5-36}$$

and, it can be shown that $|C_{33}|$ reaches a maximum when the errors are uncorrelated, i.e., $\rho_{\xi\eta} = \rho_{\xi\zeta} = \rho_{\eta\zeta} = 0$.

6.0 PROBABILITY OF HITTING A TARGET WINDOW

Mission success is generally defined in terms of "hitting" a specified target or "window". For example, in the case of a ballistic missile, we are concerned with the probability of impact within a specific target area. In order to determine this probability, it is necessary to integrate an appropriate probability density function over the area.

Alternatively, the boundaries of this target impact area may [illegible] in terms of a position and velocity "window" at "cut-off". For space vehicle missions, in general, this concept of "hitting" a prescribed "window" of position and velocity conditions is employed. The general problem of determining the probability of success is then one of integrating joint density functions over the appropriate dimensions. The boundaries for this integration will, of course, depend upon mission requirements.

In the application at hand, the probability density function is assumed to be Gaussian, and therefore the problem is one of evaluating definite integrals. Numerical procedures for the evaluation of such integrals by digital computers are presented in this section. Some typical three-dimensional "windows" have been selected and procedures are formulated for computing the probability of "hitting" these "windows". These "windows" include a sphere, cylinder, and rectangular parallelepiped, with variable dimensions; these dimensions may be chosen as either scalar components of position or scalar components of velocity.

6.1 Integration Scheme

A five-point Newton-Cotes formula (see Reference 5) is employed for the numerical integration of the definite integrals. The basic integration formula is written

$$\int_{x_0}^{x_4} g(x)\,dx = \frac{2h}{45}\left(7g_0 + 32g_1 + 12g_2 + 32g_3 + 7g_4\right) + \epsilon \qquad (6\text{-}1)$$

where $g_0 = g(x_0)$, $g_1 = g(x_1)$, $g_2 = g(x_2)$, $g_3 = g(x_3)$, and $g_4 = g(x_4)$. The coordinates x_0, x_1, x_2, x_3, and x_4 are equally spaced on the integration internal h. The error term ϵ is given in Reference 5 as follows

$$\epsilon = -\frac{8h^7}{945} \quad f^{vi} \qquad (6\text{-}2)$$

where f^{vi} is the sixth-order finite difference in the values of g_0, g_1, g_2, g_3, and g_4.

The selection of this five-point formula facilitates the use of an automatic covergence check which involves multiple passes in the integration. In each successive pass, the integration interval h is halved. Since the coordinates of the five points within the interval h are evenly spaced, resulting in an is given on page 39. The values listed in this table are computed from Equation (6-4) using the numerical integration scheme described in Section 6-1. The error standard used in this computation is ESTD = 1×10^{-7}.

6.3 Two-Dimensional Target Windows

An example of a two-dimensional target window would be the impact area of a ballistic missile. The concept of circular probable error (CPE) is generally used in this regard. The CPE is defined as the radius of a circle for which there is a 50 percent probability that the missile will impact within the circle. Alternately the CPE concept is generalized by considering the probability that the missile will impact within a circle of radius R.

In order to study this example, we may use the tracking station Cartesian coordinate system described in Reference 6. The tracking station location is regarded as a target. Scalar components of the target miss distance are measured along the u and v axes of this coordinate system, and the random variable R is defined in terms of the random variables U and V.

The joint probability density for U and V is written

$$f(u,v) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} \exp\left(-\tfrac{1}{2}\bar{X}_2^T C_{22}^{-1} \bar{X}_2\right) \tag{6-5}$$

where the subscripts 1 and 2 would refer to the u and v coordinates, respectively, in the present example. The exponential argument of Equation (6-5) is written

$$\bar{X}_2^T C_{22}^{-1} \bar{X}_2 = \frac{1}{1-\rho_{12}^2}\left[\frac{(u-m_1)^2}{\sigma_1^2} + \frac{2\rho_{12}(u-m_1)(v-m_2)}{\sigma_1\sigma_2} + \frac{(v-m_2)^2}{\sigma_2^2}\right]$$

Transforming to polar coordinates

$$u = r\cos\phi$$

$$v = r\sin\phi$$

and equating probability elements yields

$$\begin{aligned} f(u,v)\,du\,dv &= f(r,\phi)\,|J|\,dr\,d\phi \\ &= p(r,\phi)\,dr\,d\phi \end{aligned} \tag{6-6}$$

where $|J|$ is the Jacobian determinant $\frac{\partial(u,v)}{\partial(r,\phi)}$. Performing the required substitutions, the probability density, in terms of polar coordinates is written

$$\begin{aligned} p(r,\phi) = \frac{r}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} \exp\Big[& r^2k_1 + r^2k_2\cos 2\phi \\ & + r^2k_3\sin 2\phi + rk_4\cos\phi + rk_5\sin\phi + k_6\Big] \end{aligned} \tag{6-7}$$

even multiple of subintervals, the previously computed data points are used on each subsequent pass.

When the absolute rate of convergence error is less than some specified small value, called the error standard (ESTD), the final answer is obtained. The convergence error is defined as the difference between the final solution and the previous solution that occurs in the multiple pass integration process. The error standard (ESTD) is the ratio of this difference in successive passes. Previous experience with this multiple pass integration scheme indicates that the final answer should be accurate to within two or three times the specified ESTD. Too small a value for the ESTD can, however, result in excessive computer time.

To insure that pertinent features of the integrand g(x) are not missed by a large step size, a maximum step size check has been included in the integration procedure. An obvious choice for the maximum integration step size, for the present application, might be the standard deviation of the probability density function. A minimum step size check is also employed to avoid excessive computer usage.

6.2 One-Dimensional Target Windows

There exists several one-dimensional target windows of interest. For example, consider the flight path coordinate system and the system of orbital elements described in Reference 6. In the flight path coordinate system, the measurements of total velocity, flight path angle, azimuth, or local altitude are quantities which may independently provide meaningful measures of performance. Similarly, in the orbital element system, any one of the orbital elements could be the sole quantity of interest. The one-dimensional, marginal probability is written

$$f(t) = f\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \tag{6-3}$$

where t is a standardized random variable that has a zero mean and a unit standard deviation. The random variable x is the error associated with one of the measures in either the flight path coordinate system or the system of orbital elements. The mean m and the variance σ^2 of this random variable can be obtained by the propagation procedure given in Reference 6.

Tables of the probability $P(\frac{x-m}{\sigma} \le t)$ for a one-dimensional Gaussian distribution are given in most textbooks on statistics (see Reference 7, pages 209-213). A similar table, for the following integral:

$$P\left(-t < \frac{x-m}{\sigma} \le t\right) = \sqrt{\frac{2}{\pi}} \int_0^t \exp\left(-\frac{t^2}{2}\right) dt \tag{6-4}$$

TABLE [illegible]

TYPICAL COMPUTER RESULTS

ONE DIMENSIONAL GAUSSIAN INTEGRATION

t	F(t)	t	F(t)
0	0.0	2.4	0.98360490
0.2	0.15851942	2.5	0.98758065
0.4	0.31084348	2.6	0.99067760
0.6	0.45149376	2.8	0.99488971
0.8	0.57628920	3.	0.99730019
1.	0.68268948	3.2	0.99862570
1.2	0.76986066	3.4	0.99932611
1.4	0.83848667	3.6	0.99968175
1.6	0.89040139	3.8	0.99985529
1.8	0.92813935	4.	0.99993662
2.	0.95449972	5.	0.99999936
2.2	0.97219308	10.	0.99999992

where

$$k_1 = -\frac{\sigma_1^2 + \sigma_2^2}{4\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}$$

$$k = \frac{\sigma_1^2 - \sigma_2^2}{4\sigma_1^2\sigma_2^2(1-\rho_{12}^2}$$

$$k_3 = \frac{-\rho_{12}}{2\sigma_1\sigma_2(1-\rho_{12}^2)}$$

$$k_4 = \frac{\sigma_2 m_1 + \rho_{12}\sigma_1 m_2}{\sigma_1^2\sigma_2(1-\rho_{12}^2)}$$

$$k_5 = \frac{\sigma_1 m_2 + \rho_{12}\sigma_2 m_1}{\sigma_1\sigma_2^2(1-\rho_{12}^2)}$$

$$k_6 = -\frac{\sigma_2^2 m_1^2 + \sigma_1^2 m_2^2 + 2\rho_{12}\sigma_1\sigma_2 m_1 m_2}{2\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}$$

The probability that the impact errors will be within a circle of radius R is now written

$$P(0 < r \leq R) = \int_0^R \int_0^{2\pi} p(r,\phi)\, d\phi\, dr$$

$$= K\int_0^R r\, H(r) \exp(r^2 k_1)\, dr \tag{6-8}$$

where

$$K = \frac{\exp(K_6)}{2\pi\,\sigma_1\,\sigma_2\sqrt{1-\rho_{12}^2}}$$

$$H(r) = \int_0^{2\pi} \exp\left(r^2 k_2 \cos 2\phi + r^2 k_3 \sin 2\phi + r k_4 \cos\phi + r k_5 \sin\phi\right) d\phi$$

A digital computer program has been implemented for the computation of Equation (6-8). Typical results obtained from this program are presented in Table II, page 42. A value of 1×10^{-7} for the ESTD was used in these computer runs, and the computer time per case was about one minute.

If one considers only the 50 percent probability definition of the CPE, the results presented in Reference 8, page 473, may be used to avoid the expense of a digital computer program. A curve of CPE/ σ_2 versus σ_1 / σ_2 is given in this reference and a linear approximation relating the CPE to σ_2 and σ_1 is established. These results are valid, however, only when u and v are uncorrelated. No attempt is made in the present study to extend the simplified results of Reference 8 for correlated variables and other probabilities. This extension could be accomplished by using results obtainable from the digital computer program.

TABLE [illegible]

TYPICAL COMPUTER RESULTS

PROBABILITY THAT THE ERRORS ARE WITHIN A CIRCLE OF RADIUS R

ρ_{12}	σ_1,	σ_2,	m_1,	m_2	Radius	Probability
0.5	4	3	2	1	1	[illegible]
					2	0.1521466
					8	0.8732792
					10	0.95095075
	1	1	0	0	1.	0.3934693
					1.1774	0.4999941
					1.665	0.7499545
					2.	0.8646646
					2.4477	0.9499942
					3.	0.9888909
					10.	0.9999998
0	1	1	2	0	1	0.0818923
					2	0.3964990
					3	0.7856377
					4	0.9658649
					6	0.9999432
					10	0.9999996
0	1	0.4	2	3	3	0.12179[illegible]8
					5	0.9693474

6.4 The Three-Dimensional Integrations

It has been pointed out that we are concerned with the probability of "hitting" three dimensional "windows" of either position or velocity errors. The specific cases considered here are

(1) a sphere

(2) a cylinder

(3) a rectangular parallelepiped

The probability density in all cases is assumed to be a three-dimensional Gaussian distribution centered at a non-zero mean. This distribution may be regarded as either a marginal distribution of position errors without regard for velocity errors or a marginal distribution of velocity errors without regard for position errors. It is also assumed that the errors are associated with measurements in a Cartesian reference frame.

The following derivations are applicable to either a marginal distribution of position errors or a marginal distribution of velocity errors. For convenience, the notation is generalized by the use of subscript indices; the subscripts (1, 2, 3) will refer to either position errors (ξ, η, ζ) or velocity errors ($\dot{\xi}, \dot{\eta}, \dot{\zeta}$), respectively.

From Equation (2-25), the probability density is written

$$f(\bar{X}_3) = \frac{1}{\sqrt{(2\pi)^3 \, |C_{33}|}} \exp\left(-\frac{1}{2} \bar{X}_3^T C_{33}^{-1} \bar{X}_3\right) \tag{6-9}$$

where $\bar{x}_3$ is a column vector $(x_1, x_2, x_3)^T$ of the errors about their mean values.

Specifically,

$$x_1 = \epsilon_1 - m_1$$

$$x_2 = \epsilon_2 - m_2$$

$$x_3 = \epsilon_3 - m_3$$

where ($\epsilon_1, \epsilon_2, \epsilon_3$) denote either (ξ, η, ζ) or ($\dot{\xi}, \dot{\eta}, \dot{\zeta}$). If position errors are being considered, C_{33} is the upper left quadrant of (2-22); alternately, if velocity errors are being considered, C_{33} is the lower right quadrant of (2-22).

Expanding the exponential argument of (6-9)

$$-\frac{1}{2} \bar{X}_3^T C_{33}^{-1} \bar{X}_3 = k_1 \epsilon_1^2 + k_2 \epsilon_2^2 + k_3 \epsilon_3^2 + k_4 \epsilon_1 \epsilon_2 + k_5 \epsilon_1 \epsilon_3 + k_6 \epsilon_2 \epsilon_3 + k_7 \epsilon_1 + k_8 \epsilon_2 + k_9 \epsilon_3 + k_{10} \tag{6-10}$$

where

$$k_1 = \frac{\sigma_{23}^2 - \sigma_2^2\sigma_3^2}{2\,|C_{33}|}\,; \qquad k_2 = \frac{\sigma_{13}^2 - \sigma_1^2\sigma_3^2}{2\,|C_{33}|}\,;$$

$$k_3 = \frac{\sigma_{12}^2 - \sigma_1^2\sigma_2^2}{2\,|C_{33}|}$$

$$k_4 = \frac{\sigma_{12}\sigma_3^2 - \sigma_{13}\sigma_{23}}{|C_{33}|}\,; \qquad k_5 = \frac{\sigma_{13}\sigma_2^2 - \sigma_{12}\sigma_{23}}{|C_{33}|}\,;$$

$$k_6 = \frac{\sigma_{23}\sigma_1^2 - \sigma_{13}\sigma_{12}}{|C_{33}|}\,;$$

$$k_7 = \frac{1}{|C_{33}|}\Big[(\sigma_2^2\sigma_3^2 - \sigma_{23}^2)\,m_1 + (\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_3^2)\,m_2 + (\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_2^2)\,m_3\Big]\,;$$

$$k_8 = \frac{1}{|C_{33}|}\Big[(\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_3^2)\,m_1 + (\sigma_1^2\sigma_3^2 - \sigma_{13}^2)\,m_2 + (\sigma_{13}\sigma_{12} - \sigma_1^2\sigma_{23})\,m_3\Big]\,;$$

$$k_9 = \frac{1}{|C_{33}|}\Big[(\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_2^2)\,m_1 + (\sigma_{13}\sigma_{12} - \sigma_1^2\sigma_{23})\,m_2 + (\sigma_1^2\sigma_2^2 - \sigma_{12}^2)\,m_3\Big]\,;$$

and

$$k_{10} = -\tfrac{1}{2}\,(k_7 m_1 + k_8 m_2 + k_9 m_3)$$

6.4.1 The Target Sphere

The familiar circular probable error (CPE) concept has been discussed in Section 6.3. It is now convenient to generalize this concept to a spherical probable error (SPE). Consider a sphere of radius R centered at (a_1, a_2, a_3) in the $\epsilon_1, \epsilon_2, \epsilon_3$ coordinate system, as shown in Figure (6-1).

It is convenient to transform to spherical coordinates,

$$y_1 = r \sin \Psi$$

$$y_2 = r \sin\theta \cos\Psi$$

$$y_3 = r \cos\theta \cos\Psi$$

where y_1, y_2, y_3 are rectangular coordinates aligned parallel to the $\epsilon_1, \epsilon_2, \epsilon_3$ coordinates, but with the origin translated to the sphere's center. The x_1, x_2, x_3 coordinates of Equation (6-9) are then written

$$X_1 = r \sin \Psi - (m_1 - a_1)$$

$$X_2 = r \sin \theta \cos \Psi - (m_2 - a_2)$$

$$X_3 = r \cos \theta \cos \Psi - (m_3 - a_3)$$

and the exponential argument of Equation (6-9) becomes

$$\begin{aligned} -\tfrac{1}{2} \bar{X}_3^T C_{33}^{-1} \bar{X}_3 = {} & r^2 \Big[K_1 \sin^2 \Psi + K_2 \sin^2 \theta \cos^2 \Psi + K_3 \cos^2 \theta \cos^2 \Psi \\ & + K_4 \sin \theta \sin \Psi \cos \Psi + K_5 \cos \theta \sin \Psi \cos \Psi \\ & + K_6 \sin \theta \cos \theta \cos^2 \Psi \Big] + r \Big[K_7 \sin \Psi \\ & + K_8 \sin \theta \cos \Psi + K_9 \cos \theta \cos \Psi \Big] + K_{10} \end{aligned} \qquad (6\text{-}11)$$

where $K_1, K_2, \ldots, K_{10}$ are obtained from Equation (6-10) by replacing m_1 with $m_1 - a_1$, m_2 with $m_2 - a_2$, and m_3 with $m_3 - a_3$. The probability that the errors are within a target sphere of radius R is then,

$$\begin{aligned} P(0 < r \leq R) = {} & \frac{\exp(K_{10})}{\sqrt{(2\pi)^3 \, |C_{33}|}} \int_0^R r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \Psi \exp(K_1 r^2 \sin^2 \Psi + K_7 r \sin \Psi) \\ & \times \int_0^{2\pi} \exp\Big[r^2 \cos^2 \Psi (K_2 \sin^2 \theta + K_3 \cos^2 \theta + K_6 \sin \theta \cos \theta) \\ & + r^2 \sin \Psi \cos \Psi (K_4 \sin \theta + K_5 \cos \theta) \\ & + r \cos \Psi (K_8 \sin \theta + K_9 \cos \theta) \Big] \, d\theta \, d\Psi \, dr \end{aligned} \qquad (6\text{-}12)$$

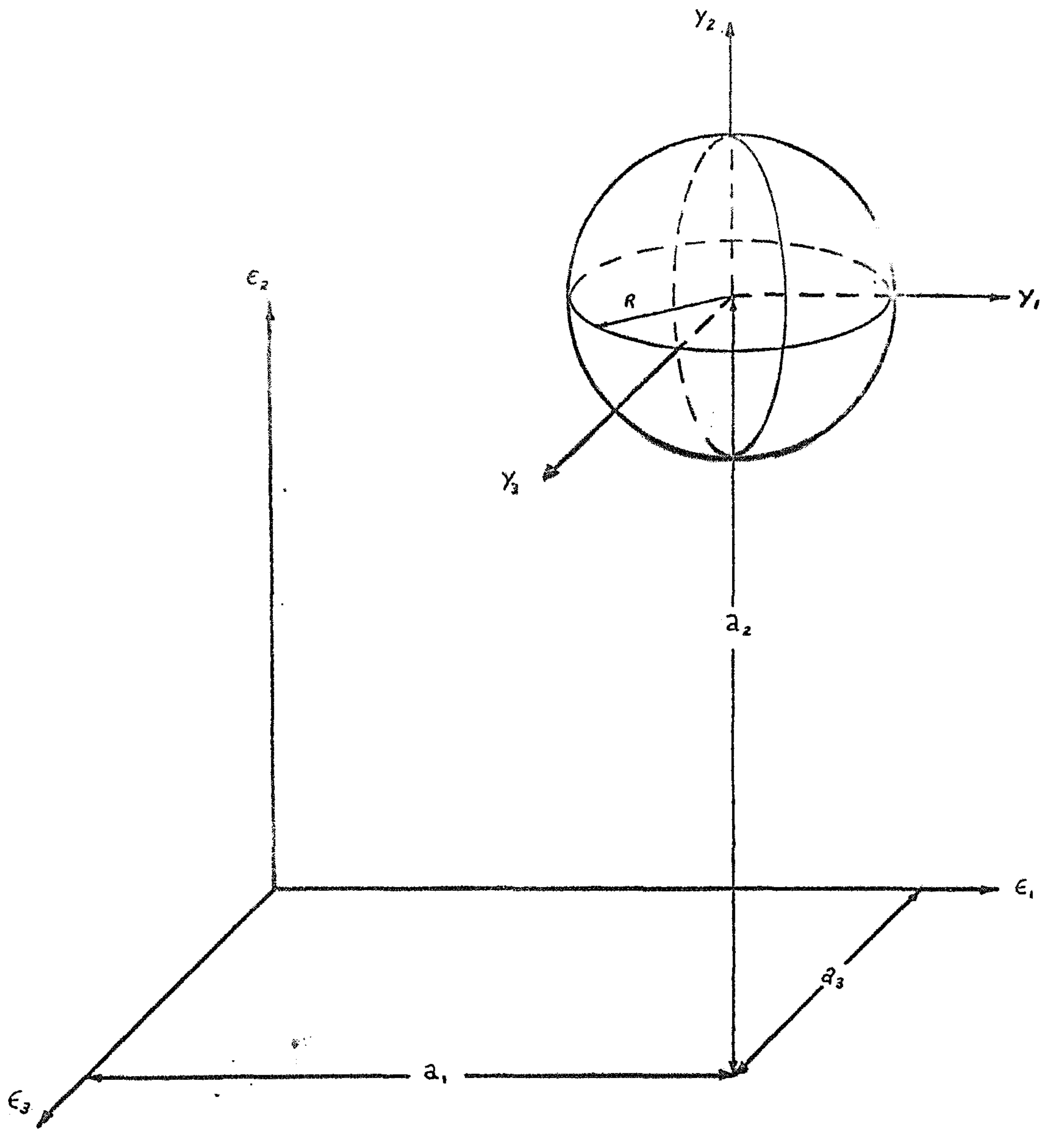


Figure 6-1 - A Spherical Error Window (Target Sphere)

A digital computer program based upon the formulation of Equation (5-12) has been implemented. Typical results of the computation are presented in Table III, page 48. The computer time required for these computations is rather excessive, and in some cases it could be prohibitive. For an error convergence standard (ESTD) of 1×10^{-4}, approximately four minutes of computer time per case was required.

These computer results indicate that a more sophisticated integration scheme is required to reduce computer run time. A scheme which automatically reduces the step size around the peaks of the distribution and increases the step size in the tails of the distribution might be considered.

TABLE III

TYPICAL COMPUTER RESULTS

PROBABILITY THAT THE ERRORS ARE WITHIN A TARGET SPHERE

Means (m_1, m_2, m_3)	Covariance Matrix	Radius of sphere	Probability	Convergence Error*
$(0, 0, 0)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1	0.19874801	-0.937×10^{-8}
		2	0.73853579	0.807×10^{-7}
		3	0.97070900	-0.230×10^{-7}
		4	0.99886590	-0.24[illegible] $\times 10^{-7}$
		10	0.99999986	-0.298×10^{-7}

*See Section 6.1

6.4.2 The Target Cylinder

Consider a cylinder of radius R, centered at (a_1, a_2, a_3) in the $\epsilon_1, \epsilon_2, \epsilon_3$ coordinate system, as shown in Figure (6-2). The cylinder is oriented here such that its figure axis is parallel to the ϵ_3 coordinate. This is not a restriction on the orientation of the actual target; the covariance propagation technique described in Reference 6 may be used to define the means and covariances in an $\epsilon_1, \epsilon_2, \epsilon_3$ system compatible with any prescribed orientation.

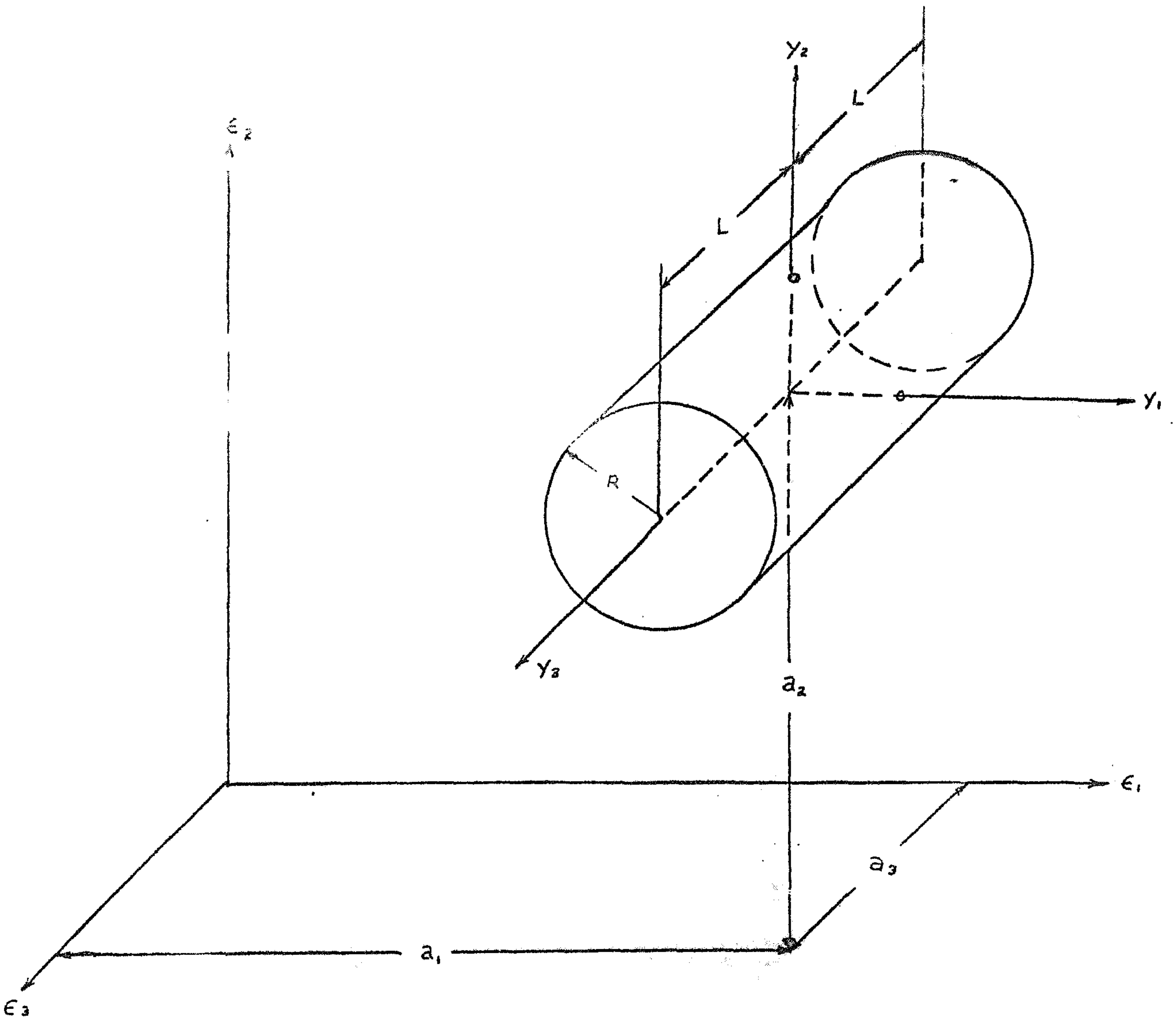


Figure 6-2 - A Cylindrical Error Volume (Target Cylinder)

It is convenient to transform to cylindrical coordinates, where

$$y_1 = r \sin \psi$$
$$y_2 = r \cos \psi \qquad (6\text{-}13)$$
$$y_3 = y_3$$

The exponential argument of Equation (6-9) is then written

$$-\frac{1}{2}\bar{X}_3^T C_{33}^{-1}\bar{X}_3 = K_1 r^2 \sin^2\psi + K_2 r^2 \cos^2\psi + K_3 \epsilon_3^2$$
$$+ K_4 r^2 \sin\psi \cos\psi + K_5 \epsilon_3 r \sin\psi \qquad (6\text{-}14)$$
$$+ K_6 \epsilon_3 r \cos\psi + K_7 r \sin\psi$$
$$+ K_8 r \cos\psi + K_9 \epsilon_3 + K_{10}$$

where K_1, K_2, ..., K_{10} are again obtained from Equation (6-10) by replacing m_1 with $m_1 - a_1$, m_2 with $m_2 - a_2$, and m_3 with $m_3 - a_3$.

The probability that the errors are contained within the prescribed cylinder is written

$$P(0 < r \le R, -\ell < \epsilon_3 \le \ell) = \frac{\exp(K_{10})}{\sqrt{(2\pi)^3 |C_{33}|}} \int_{-\ell}^{\ell} \exp(K_9 \epsilon_3 + K_3 \epsilon_3^2)$$
$$\times \int_0^R r \int_0^{2\pi} \exp\Big[r^2(K_1 \sin^2\psi + K_2 \cos^2\psi + K_4 \sin\psi \cos\psi) \qquad (6\text{-}15)$$
$$+ r(K_7 \sin\psi + K_8 \cos\psi) + \epsilon_3 r(K_5 \sin\psi + K_6 \cos\psi)\Big] \, d\psi \, dr \, d\epsilon_3$$

A computer program based upon the formulation of Equation (6-15) has been implemented. Typical results of the computation are presented in Table IV, page 51. A convergence error standard (ESTD) of 1 x 10^{-7} was used to obtain these results.

TABLE IV

TYPICAL COMPUTER RESULTS

PROBABILITY THAT THE ERRORS ARE WITHIN A TARGET CYLINDER

Means (m_1, m_2, m_3)	Covariance Matrix	Cylinder Radius	Cylinder Semi-length	Probability	Convergence Error*
(0, 0, 0)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	1	1	0.26861735	0.139×10^{-7}
		5	1	0.68268680	0.109×10^{-7}
		5	5	0.99999547	-0.22×10^{-7}

*See Section 6.1

6.4.3 The Rectangular Parallelepiped

The orientation of the rectangular parallelepiped is restricted such that the sides are parallel to the axis system associated with the covariance matrix of errors. This restriction may be circumvented by simply propagating the covariances and means into the appropriate axis system. The dimensions of this box are shown in Figure (6-3), where L_i is the lower limit of the box and U_i is the upper limit of the box in the ϵ_i dimension.

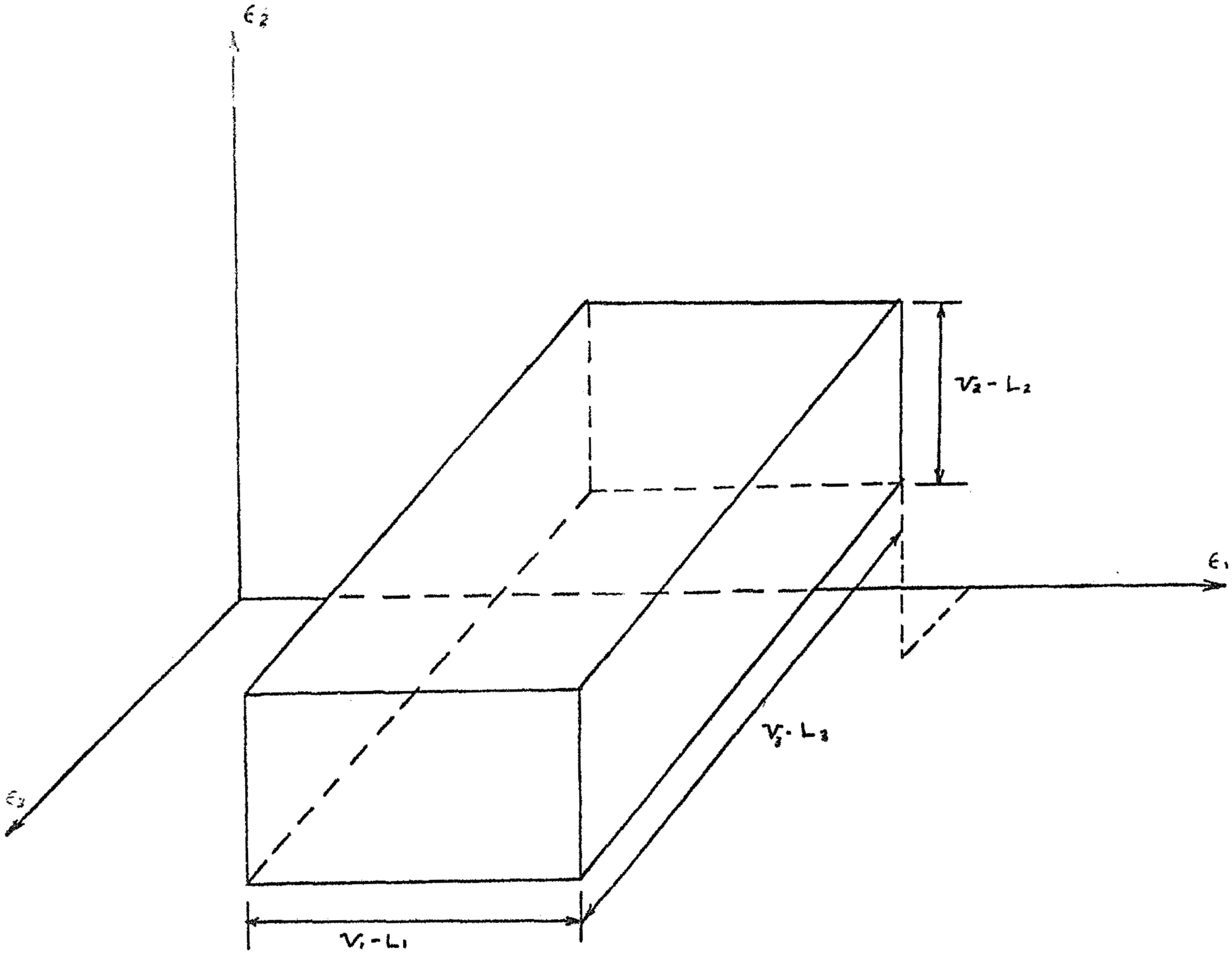


Figure 6-3 - A Rectangular Parallelepiped Error Window (Target Box)

The probability that the errors will be contained in the prescribed box is written

$$P(L_1 < \epsilon_1 \leq V_1 , L_2 < \epsilon_2 \leq V_2 , L_3 < \epsilon_3 \leq V_3)$$

$$= \frac{\exp (K_{10})}{\sqrt{(2\pi)^3 |C_{33}|}} \int_{L_1}^{V_1} \exp (K_1 \epsilon_1^2 + K_7 \epsilon_1) \int_{L_2}^{V_2} \exp (K_2 \epsilon_2^2 + K_4 \epsilon_1 \epsilon_2 + K_8 \epsilon_2) \tag{6-16}$$

$$\times \int_{L_3}^{V_3} (K_3 \epsilon_3^2 + K_5 \epsilon_1 \epsilon_3 + K_6 \epsilon_2 \epsilon_3 + K_9 \epsilon_3) \, d\epsilon_1 \, d\epsilon_2 \, d\epsilon_3$$

A computer program has been formulated for the evaluation of Equation (6-16). Typical results of the computation are presented in Table V, page 54. A convergence error standard (ESTD) of 1×10^{-7} was used to obtain these results.

TABLE V

TYPICAL COMPUTER RESULTS

PROBABILITY THAT THE ERRORS ARE WITHIN A RECTANGULAR PARALLELEPIPED

Means (m_1, m_2, m_3)	Covariance Matrix	Dimensions $(L_1, U_1, L_2, U_2, L_3, U_3)$	Probability	Convergence Error
(0, 0, 0)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(-1, 1, -1, 1, -1, 1)	0.31817759	0.23[illegible] $\times 10^{-7}$
		(-5, 5, -5, 5, -5, 5)	0.99999803	-0.119×10^{-7}
		(-1, 2, -3, 4, -5, 6)	0.81746322	0.456×10^{-7}

*See Section 6.1

7.0 SIMPLIFIED MEASURES OF PERFORMANCE

The previous two chapters have presented several formulations to obtain cumulative probability levels associated with various target windows or error volumes. The equiprobability ellipsoid, by virtue of its specialization, was capable of an analytic solution whereas the more general cases of two-dimensional circles and three-dimensional spheres, cylinders, and boxes must be solved numerically using the digital computer. However, for preliminary estimates and simplified analyses, there are many simplified performance measures that are quite useful.

In 6.3 and 6.4.1, the concepts of CPE (Circular Probable Error) and SPE (Spherical Probable Error) were introduced. It will be recalled that these measures give the radius of a circle or sphere such that the cumulative probability of hitting these windows is 50 percent.

Instead of basing the equivalence on the cumulative probability level, the equivalence could be based on the error volume content. In this way the radius of a circle or sphere of identical volume content could be determined. The general equation for the volume of an n-dimensional sphere is given by Reference [illegible] as:

$$V_n = \frac{\pi^{n/2} r^n}{(n/2)!} \tag{7-1}$$

and, from 5.3, the volume of the equiprobability ellipsoid is

$$V_n = \frac{(\pi K)^{n/2}\sqrt{|C_{nn}|}}{(n/2)!} \tag{5-32}$$

Equating the two volumes

$$r = \sqrt{K}\,|C_{nn}|^{1/2n} \qquad n = 2, 3, 4, \cdots \tag{7-2}$$

For the probability level of interest, k can be read from Figure (5.1).

The ellipsoidal error volume is directly proportional to the square root of $|C_{nn}|$, Equation (5-32). This quantity is termed the generalized variance and is simply the product of the eigenvalues

$$|C_{nn}| = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n = \sigma_{\xi'}^2 \sigma_{\eta'}^2 \sigma_{\zeta'}^2 \cdots \tag{7-3}$$

If it were convenient to do so, an equivalent variance or standard deviation could be determined for any covariance matrix C_{nn}. We have

$$\sigma_{eq}^2 = |C_{nn}|^{1/n}$$

$$\sigma_{eq} = |C_{nn}|^{1/2n} \tag{7-4}$$

In this way, a general notion of the spread associated with the n-dimensional distribution could be obtained.

A very useful approximation exists for $|C_{nn}|$. By Equation (2-22a),

$$|C_{nn}| = \sigma_1^2 \sigma_2^2 \cdots |\rho_{nn}| \tag{7-5}$$

where $|\rho_{nn}|$ is the determinant of the correlation matrix.

Since error volumes, in general, are proportional to the square root of the generalized variance, an easy way to approximate this quantity is sought. By neglecting the covariance terms, $|C_{nn}| \sim \sigma_1^2 \sigma_2^2 \cdots\cdots$, and, if it can be shown that $|C_{nn}| \leq \sigma_1^2 \sigma_2^2 \cdots$ the approximate error volume will always be larger than the actual volume. In this way, if mission specifications are met with the approximation, they will be met to an even higher degree by considering the covariance elements. Hence, the approximation is termed conservative.

Thus, we seek to prove that

$$|C_{nn}| \leq \sigma_1^2 \sigma_2^2 \cdots = \prod_{i=1}^{n} \sigma_i^2 \tag{7-6}$$

Referring to (7-5), this reduces to the requirement that

$$|\rho_{nn}| \leq 1 \tag{7-7}$$

Now the correlation matrix ρ_{nn} is an n x n symmetric matrix. Any matrix of this type can be diagonalized by an orthogonal transformation of the form $\Phi^T \rho_{nn} \Phi$ where Φ is the eigenvector matrix formed with the eigenvectors as successive columns. Each eigenvector $\vec{\phi}_i$ is associated with an eigenvalue λ_i. Thus

$$\tilde{\rho}_{nn} = \Phi^T \rho_{nn} \Phi = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \tag{7-8}$$

Under this type of transformation, the trace and determinant are invariant. Hence

$$\lambda_1 + \lambda_2 + \cdots\cdots + \lambda_n = \sum_{i=1}^{n} \lambda_i = n \tag{7-9}$$

and

$$|\dot{\rho}_{nn}| = |\rho_{nn}| = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i \tag{7-10}$$

Now we seek the eigenvalues such that $|\rho_{nn}|$ is maximum. Equation (7-9) is introduced into (7-10) yielding

$$|\rho_{nn}| = \prod_{i=1}^{n-1} \lambda_i \left(n - \sum_{i=1}^{n-1} \lambda_i \right) \tag{7-11}$$

and the partial derivatives are formed

$$\frac{\partial |\rho_{nn}|}{\partial \lambda_j} = 0 \quad ; \qquad j = 1, 2, \cdots, n-1 \tag{7-12}$$

we obtain, since all the eigenvalues must be positive,

$$n - \sum_{\substack{i=1 \\ i \neq j}}^{n-1} \lambda_i - 2\lambda_j = 0 \quad ; \qquad j = 1, 2, \cdots, n-1 \tag{7-13}$$

The condition for a maximum is $\dfrac{\partial^2 |\rho_{nn}|}{\partial \lambda_j^2} < 0$. In this case

$$\frac{\partial^2 |\rho_{nn}|}{\partial \lambda_j^2} = -2 \prod_{\substack{i=1 \\ (i \neq j)}}^{n-1} \lambda_i \quad ; \qquad j = 1, 2, \cdots h-1 \tag{7-14}$$

and the condition is always satisfied.

Setting j = 1, Equation (7-13) becomes

$$2\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-1} = n \tag{7-15}$$

and, subtracting the equations for j = 2, 3, . . ., n - 1 from (7-15), we obtain

$$\lambda_1 - \lambda_2 = 0$$

$$\lambda_1 - \lambda_3 = 0 \qquad (7\text{-}16)$$

$$\vdots$$

$$\lambda_1 - \lambda_{n-1} = 0$$

Hence $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 1$ and, using (7-9), $\lambda_n = 1$. Thus, by (7-10), the maximum $|\rho_{nn}|$ is unity; this is obviously the case only when all $\rho_{ij} = 0$. In all other cases when $\rho_{ij} \neq 0$, $|\rho_{nn}| < 1$. Thus (7-7) has been proven.

An error volume ratio can be defined as $\left[|C_{nn}| \Big/ \prod_{j=1}^{n} \sigma_j^2 \right]^{1/2}$; using Equation (7-5) this is equivalent to $|\rho_{nn}|^{\frac{1}{2}}$ and is simply the ratio of error volumes calculated with and without the covariance elements included. Figure (7.1) is a plot of this ratio versus the correlation coefficient where, for simplicity, all correlation coefficients are assumed equal. Dimensions from 2 to 6 are shown and it is observed that as n increases, the approximation indeed becomes more conservative.

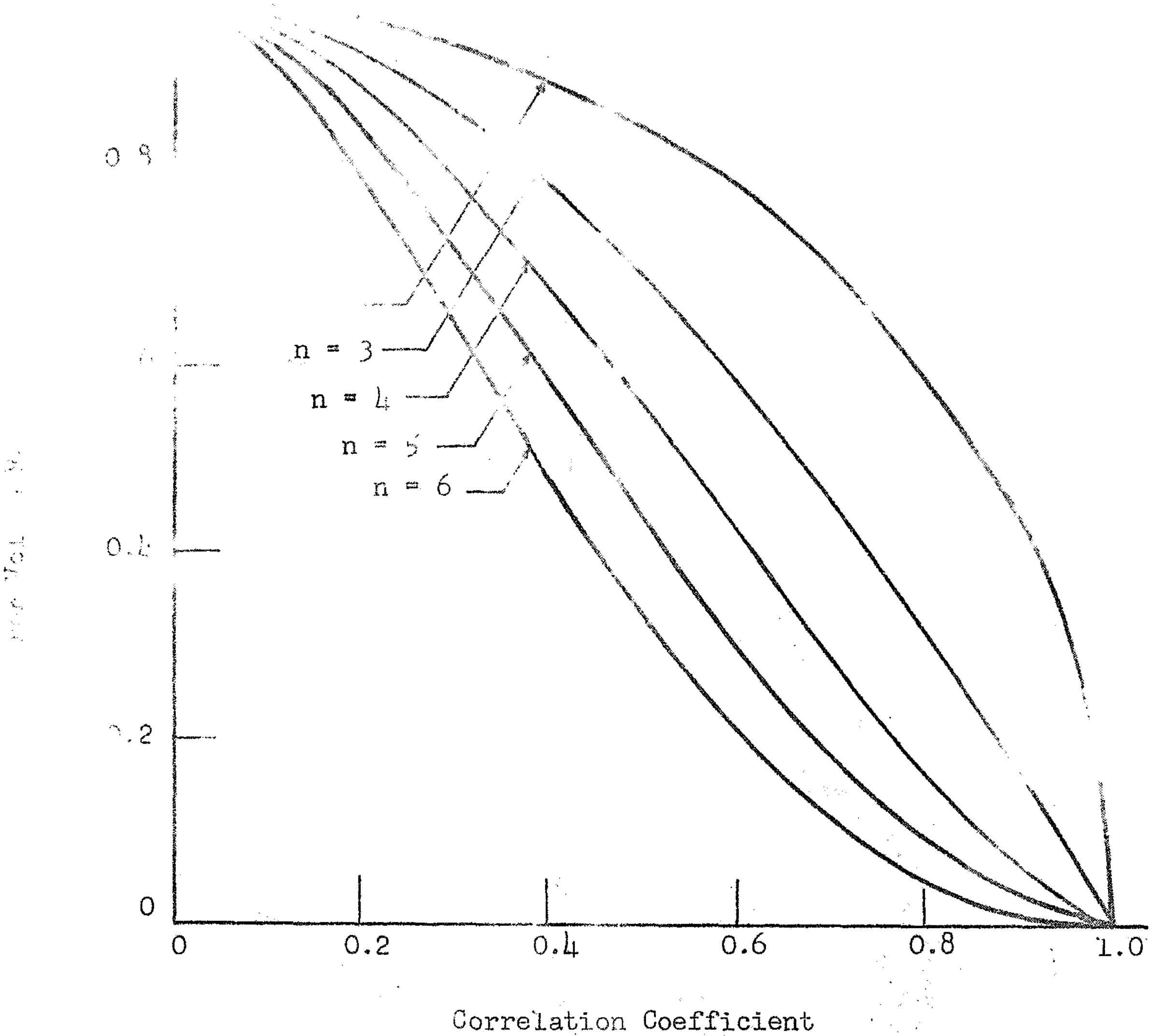


Figure 7.1 - Error Volume Ratio as a Function of Correlation Coefficient

8.0 CONCLUDING REMARKS

The formulation of the mathematical procedures for determining certain statistical measures of performance has been presented. The concepts involved have been elucidated so that these measures may be applied with an accurate understanding of their meaning.

The emphasis has been placed upon marginal distributions. For example the three dimensional error volumes and target windows are applicable to studies where one is concerned with position errors without regard to the error that may occur in the velocity measurement, or vice versa. An extension of the present formulations might involve a consideration of "conditional" error volumes. For example, consider the problem where one is concerned with the position error volume subject to the constraint that the velocity error volume be no larger than a prescribed value. Another formulation that could be included is a procedure for optimizing the error volume with respect to tolerances on the error sources.

The procedures formulated for computing the probability of hitting a target window can be improved for more efficient computer usage. A more efficient numerical integration process, tailored to the problem at hand, would enhance the accuracy of the computations and reduce the computer time.

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SPOT DIAGRAMS — FOCUS SHIFT STUDY $\lambda 1450$ Å

GRATING — $R = 200$ cm, $\alpha = 11.9869°$, $\beta_{1450} =$

$\frac{1}{d} = 2160$ ℓ/mm, $R_{s\,\lambda 1450} =$

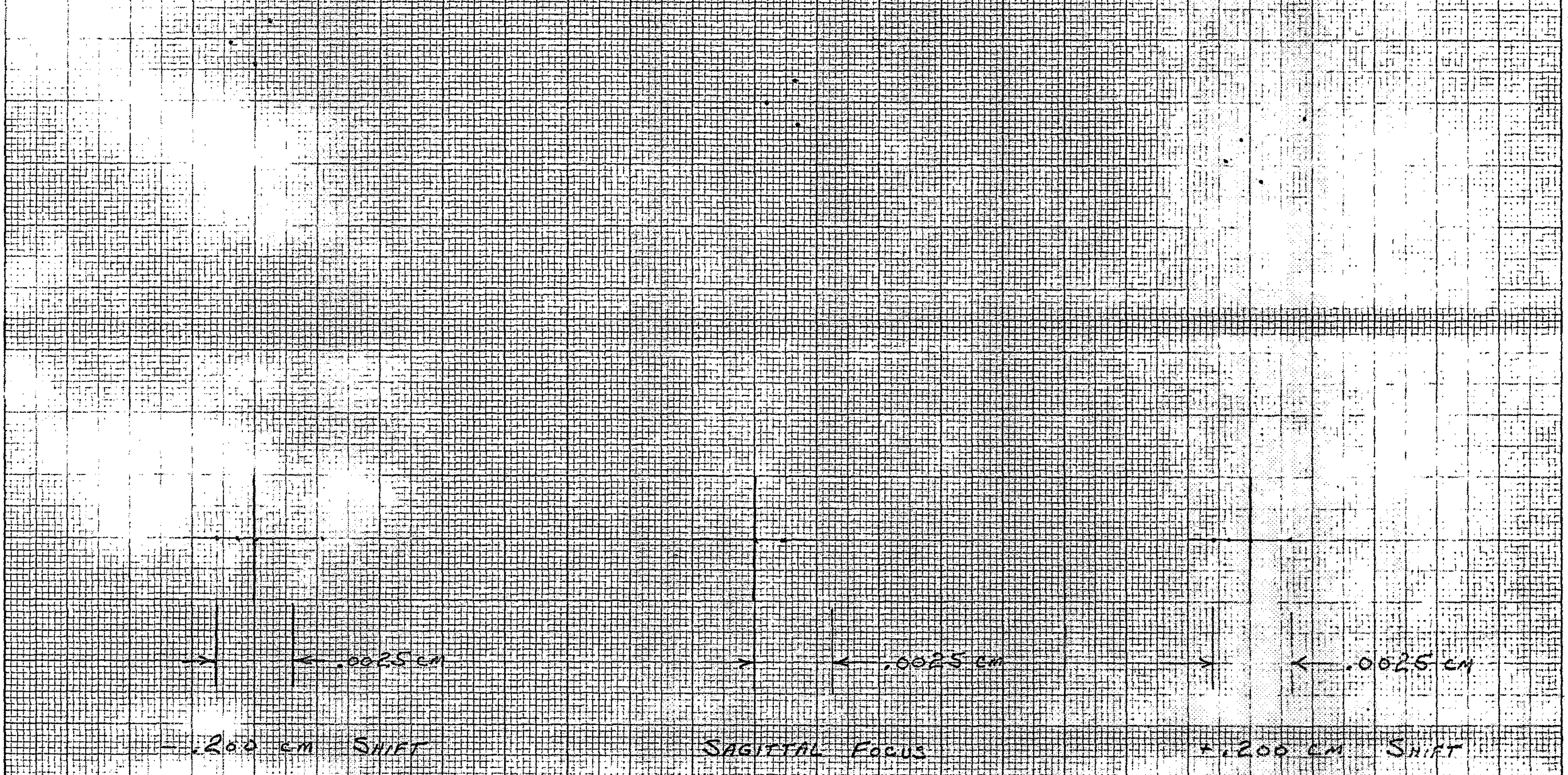

FIGURE 6

REDUCTION OF ASTIGMATISM IN ZERO ORDER IMAGE

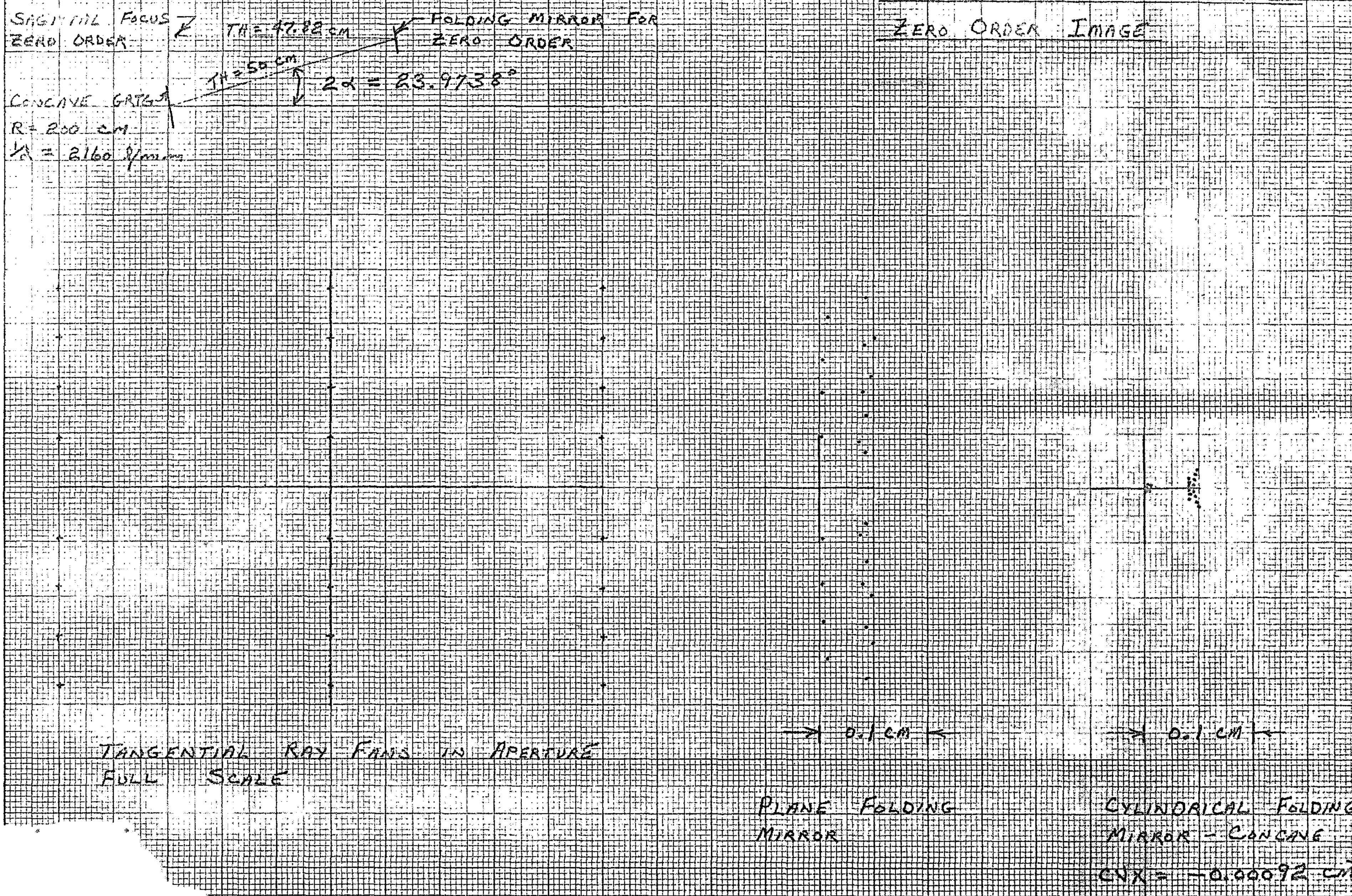

FIGURE 7